Lecture 7: Correlation + Regression

→ correlation does NOT imply causation

- two variables may be positively correlated because they're influenced by a third variable
- but here we'll consider the case of two variables

→ scatter plot

- each point represents a pair of x & y values
- we're not connecting the points

→ correlation coefficient

- $$r_{xy} = \frac{cov(x,y)}{\sqrt{var(x)}\cdot\sqrt{var(y)}} = \frac{\sum (x_i-\bar{x})(y_i-\bar{y})}{\sqrt{\sum(x_i-\bar{x})^2}\sqrt{\sum(y_i-\bar{y})^2}}$$

- Note: if $y_i = \alpha x_i \ \forall i$ then $r_{xy} = sign(\alpha)\cdot 1$

→ note that correlation coefficient may be zero if relationship is non-linear

The Gauss-Markov Assumptions

$$y = \alpha + \beta x + \varepsilon$$

1. $E[\varepsilon] = 0$

   expected value of the error term equals zero

2. $E[\varepsilon^2] = \sigma^2 < \infty$

   constant + finite variance of error term

3. $cor(\varepsilon, x) = 0$ ← zero correlation between error + regressors

4. $cor(\varepsilon_i, \varepsilon_j) = 0$ ← zero correlation among error terms

5. $cor(x_A, x_B) = 0$ ← zero correlation among regressors

6. **linear model**

   - a percentage should not be dependent variable because

$$0 < pct < 1$$

Minimizing Sum of Squares

$$y = \alpha + \beta x + \varepsilon$$

$$\varepsilon = y - \alpha - \beta x$$

$$\underset{\alpha,\beta}{\text{Min}} \quad \Sigma \varepsilon^2 = \Sigma (y - \alpha - \beta x)^2$$

$$\frac{\partial \Sigma \varepsilon^2}{\partial \alpha} = -2 \Sigma (y - \alpha - \beta x) = 0$$

necessary condition #1

two implications:

a. $\frac{1}{N} \Sigma \varepsilon = 0 \iff E[\varepsilon] = 0$

b. $\bar{y} = \hat{\alpha} + \hat{\beta} \bar{x} \implies \hat{\alpha} = \bar{y} - \hat{\beta} \bar{x}$

$$\frac{\partial \Sigma \varepsilon^2}{\partial \beta} = -2 \Sigma (y - \alpha - \beta x) x = 0$$

necessary condition #2

two implications:

a. $\frac{1}{N} \Sigma \varepsilon x = 0 \iff cov(\varepsilon, x) = 0$

note: $cov(\varepsilon, x) = \frac{1}{N} \Sigma \varepsilon x - \bar{\varepsilon} \bar{x}$

but $\bar{\varepsilon} = 0 \quad \therefore \quad cov(\varepsilon, x) = 0$

b. $$\hat{\beta} = \frac{\frac{1}{N} \Sigma y x - \bar{y} \bar{x}}{\frac{1}{N} \Sigma x^2 - \bar{x}^2} = \frac{\Sigma (y - \bar{y})(x - \bar{x})}{\Sigma (x - \bar{x})^2} = \frac{cov(x, y)}{var(x)}$$

```
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> ## quick R script to illustrate correlation and regression
>
> ## randomly generate X and Y with positive relation
> xx <- rnorm(30)
> yy <- 2*xx + rnorm(30)
>
> ## correlation
> cor(xx,yy)
[1] 0.8966973
>
> ## regression
> ols <- lm(yy~xx)
>
> ## plot the relation and
> ## insert regression line into plot
> png( "scatterplot.png" )
> plot( xx ,yy , axes = FALSE )
> axis(1 , pos = 0 )
> axis(2 , pos = 0 )
> abline( a = coef(ols)[1] , b = coef(ols)[2] )
> dev.off()
null device
          1
>
> ## how did we get coefficients?
> ## we minimized the sum of squares
> beta <- seq( 0.5 , 3.5 , 0.01 )
> sqerr <- NA
> for (i in 1:length(beta)) {
+   alfa <- mean(yy) - beta[i]*mean(xx)
+   sqerr[i] <- sum((yy - alfa - beta[i] * xx)^2)
+ }
>
> ## show sum of squares plotted over different possible values of beta
> png( "min-sum-squares.png" )
> plot( beta , sqerr , type = "l" , axes = FALSE )
> axis(1)
> axis(2)
> dev.off()
null device
          1
>
```
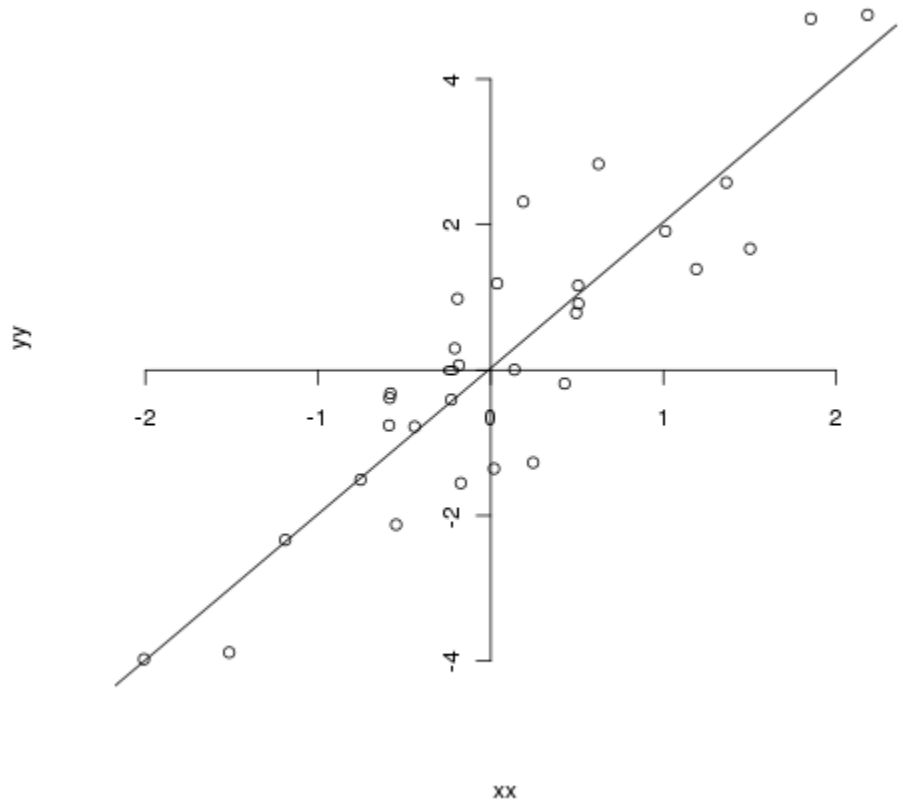

```
> ## find beta which minimizes sum of squares
> betahat <- beta[which(sqerr == min(sqerr))]
>
> ## corresponding alpha
> alfahat <- mean(yy) - betahat*mean(xx)
>
> ## compare coefficients with regression output
> alfahat
[1] 0.02580671
> betahat
[1] 2.01
>
> ## summarize regression
> summary(ols)

Call:
lm(formula = yy ~ xx)

Residuals:
     Min       1Q   Median       3Q      Max
-1.79051 -0.72444  0.02788  0.63851  1.91080

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.02611    0.17507   0.149    0.883
xx           2.00706    0.18724  10.719 2.03e-11 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.953 on 28 degrees of freedom
Multiple R-squared: 0.8041,     Adjusted R-squared: 0.7971
F-statistic: 114.9 on 1 and 28 DF,  p-value: 2.032e-11

>
> ## the R-squared:
> r2 <- 1 - (var(ols$residuals)/var(yy))
> r2
[1] 0.804066
>
> ## square root of R-squared is equal to the correlation coefficient
> sqrt(r2)
[1] 0.8966973
> cor(xx,yy)
[1] 0.8966973
>
> proc.time()
   user  system elapsed
  0.308   0.040   0.333
```

scatterplot and regression line



sum of squares as a function of “beta