Lecture 3 Probability, the Binomial Distribution + the Normal Distribution

→ previously we studied "what happened?"

→ here we'll study "what might happen?"

→ later we'll compare what happened to the probability of its occurrence

probability

→ $S \equiv$ set of possible outcomes

→ $A, B, \ldots \equiv$ events (subsets of sample space)

→ probability: $P[S] = 1$

$$0 \leq P[A] \leq 1$$

if $A + B$ are events that do not have any outcomes in common (e.g. roll of a die cannot be both 1 and 2), then the events are disjoint

If $A + B$ are disjoint events,

then $P[A] + P[B] = P[A \text{ or } B]$ ← probability of

$= P[A \cup B]$ ← union of $A + B$

→ roll of evenly weighted six-sided di

$P[1] = 1/6$, $P[2] = 1/6$, ...

$P[\text{odd}] = 1/2$, $P[\text{even}] = 1/2$

$P[1 \cup 2 \cup 3] = 1/2$, ...

→ probability is numerical answer to the question: "What are the chances of an event occurring?"

→ flip of an evenly weighted coin

$P[H] = p$ $\quad P[T] = 1-p$ $\quad p = 1/2$

→ two coin flips

events: HH, HT, TH, TT

each event equally likely

→ roll of a pair of dice

probability of the sum

$P[2] = 1/36$ $\quad P[7] = 1/6$

$P[3] = 1/18$ $\quad P[8] = 5/36$

$P[4] = 1/12$ $\quad P[9] = 1/9$

$P[5] = 1/9$ $\quad P[10] = 1/12$

$P[6] = 5/36$ $\quad P[11] = 1/18$

$P[12] = 1/36$

→ if we know the probability of each elementary event, then we can determine the probability of any event

→ intersection of two events

$A \equiv$ 3 on red di
$B \equiv$ 5 on green di

3 on red $\cap$ 5 on green

$A \cap B$

$P[A] = \frac{1}{6}$

$P[B] = \frac{1}{6}$

$$\underbrace{P[A \cap B] = \frac{1}{36} = P[A] \cdot P[B]}_{\text{because } A + B \text{ are disjoint}}$$

→ factorials

- pull one of n balls from a jar
- pull another of the remaining $n-1$ balls
- repeat until no balls left

- the total number of orderings is:

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1$$

→ odds ratio : $\frac{P[A]}{1-P[A]}$

if $P[A] = \frac{1}{6}$, then odds are "1 in 5"

$$\frac{\frac{1}{6}}{\frac{6}{6}-\frac{1}{6}} = \frac{1}{5}$$

& odds against are "5 to 1"

for each time A occurs, there will be 5 times that it doesn't

→ if you toss a coin 10 times, the probability of 5 heads & 5 tails is NOT 50%, it's approx. 25%

~~P[number of heads] =~~

$X \equiv$ number of heads
$n \equiv$ number of tosses
$p \equiv$ probability of heads

binomial coefficient

$$\binom{n}{x} = \frac{n!}{(n-x)!\,x!}$$

$$P[X|n,p] = \frac{n!}{(n-x)!\,x!} \cdot p^x \cdot (1-p)^{n-x}$$

$$252 \cdot 0{,}03125 \cdot 0{,}03125 = 24{,}6\%$$

→ in previous example

$$\binom{n}{x} = \frac{n!}{(n-x)!\,x!}$$ ← gives number of combinations where "heads" occurs x times out of n tosses

$$p^x (1-p)^{n-x}$$ ← gives probability of each combination

→ show that it works

number of heads	combinations that yield	probability
0	1	$1/16 = 0{,}0625$
1	4	$4/16 = 0{,}25$
2	6	$6/16 = 0{,}375$
3	4	$4/16 = 0{,}25$
4	1	$1/16 = 0{,}0625$
	16	1,—

→ Pascal's triangle

1	$1 = 2^0$
1 1	$2 = 2^1$
1 2 1	$4 = 2^2$
1 3 3 1	$8 = 2^3$
1 4 6 4 1	$16 = 2^4$
1 5 10 10 5 1	$32 = 2^5$

tosses		
0	$2^0 = 1$	← only one possible outcome (zero heads)
1	$2^1 = 2$	← two possible outcomes (H or T)
2	$2^2 = 4$	← four possible outcomes (HH, HT, TH, TT)
3	$2^3 = 8$	← eight possible outcomes:

HHH	HHT HTH THH	TTH HTT THT	TTT
3 heads	2 heads	1 heads	zero heads

tosses		
0	1	(zero heads, zero tails)
1	1 1	H or T
2	1 2 1	HH $\boxed{\text{HT TH}}$ TT
3	1 3 3 1	HHH $\boxed{\text{HHT HTH THH}}$ $\boxed{\text{TTH HTT THT}}$ TTT

$$\frac{0!}{0!\,0!}$$

$$\frac{1!}{1!\,0!} \qquad \frac{1!}{0!\,1!}$$

$$\frac{2!}{2!\,0!} \qquad \frac{2!}{1!\,1!} \qquad \frac{2!}{0!\,2!}$$

$$\frac{3!}{3!\,0!} \qquad \frac{3!}{2!\,1!} \qquad \frac{3!}{1!\,2!} \qquad \frac{3!}{0!\,3!}$$

note pattern in denominators of the binomial coefficient

$\boxed{\text{binomial coefficient}}$

$$\binom{n}{x} = \frac{n!}{(n-x)!\,x!}$$

gives number of combinations where "heads" occurs x times out of n tosses

→ but the number of combinations is not sufficient to tell us the probability that we'll toss heads 5 times in 10 flips

→ we also need the probability of each combination which depends on the probability of "heads" in a given toss

$$p^x(1-p)^{n-x}$$

→ in the case of a coin flip $p = \frac{1}{2}$

so $\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{n-x} = \left(\frac{1}{2}\right)^n = \frac{1}{2^n}$

which is the inverse of the total number of combinations

→					
number of heads	0	1	2	3	4
combinations that yield	1	4	6	4	1
probability	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

$$P[x \mid n, p] = \frac{n!}{(n-x)!\,x!}\, p^x (1-p)^{n-x}$$

→ probability of 5 heads in 10 tosses depends on probability of heads in one toss AND number of tosses

→ but it isn't always a coin flip

→ Your textbook has a good example of seeds. Suppose you have 8 seeds and the probability of each one germinating is 20%

$$prob(r=0) = \binom{8}{0} 0{,}2^0 \, 0{,}8^8 = 1 \cdot 1 \cdot 0{,}168 = 0{,}168$$

$$prob(r=1) = \binom{8}{1} 0{,}2^1 \, 0{,}8^7 = 8 \cdot 0{,}2 \cdot 0{,}210 = 0{,}336$$

$$prob(r=2) = \qquad = 0{,}293601$$

$$prob(r=3) = \qquad = 0{,}146801$$

$$prob(r=4) = \qquad = 0{,}045875$$

$$prob(r=5) = \qquad = 0{,}009175$$

$$prob(r=6) = \qquad = 0{,}001147$$

$$prob(r=7) = \qquad = 0{,}000082$$

$$prob(r=8) = \binom{8}{8} 0{,}2^8 \, 0{,}8^0 = 1 \cdot 0{,}000003 \cdot 1 = 0{,}000003$$

$$1{,}$$

$$P[X \geq 4] = 0{,}046 + 0{,}009 + 0{,}001 + 0{,}000 + 0{,}000 = 0{,}056$$

probability of obtaining 4 or more plants is approx 0,056

Expected Value + Variance

$$E[X] = \mu$$
$$= x_1 \cdot P[X = x_1] + \dots + x_m \cdot P[X = x_m]$$

where m is number of possible values of X

example: roll of a di

$$E[x] = 1 \cdot \tfrac{1}{6} + 2 \cdot \tfrac{1}{6} + 3 \cdot \tfrac{1}{6} + 4 \cdot \tfrac{1}{6} + 5 \cdot \tfrac{1}{6} + 6 \cdot \tfrac{1}{6}$$
$$= 3{,}5$$

expected value of a roll of two dice is 7

When the random variable can take only one of two possibilities the ~~sum~~ result is much simpler

$$E[X] = n \cdot p$$

Why?

$$E[X] = \sum_{i=1}^{n} \left(\underbrace{0 \cdot (1-p)}_{\text{zero}} + 1 \cdot p \right)$$

$$= np$$

$$Var[X] = E[(x-\mu)^2 P[x]]$$

$$= \sigma^2$$

$$= \sum_{i=1}^{m} (x_i - \mu)^2 \cdot P[X=x_i]$$

$$= \sum_{i=1}^{m} (x_i^2 - 2\mu x_i + \mu^2) \cdot P[X=x_i]$$

$$= \sum_{i=1}^{m} x_i^2 \cdot P[X=x_i] - 2\mu \cdot \underbrace{\sum_{i=1}^{m} x_i \cdot P[X=x_i]}_{E[X]=\mu} + \mu^2 \underbrace{\sum_{i=1}^{m} P[X=x_i]}_{\text{one}}$$

$$= \sum_{i=1}^{m} x_i^2 \cdot P[X=x_i] - 2\mu^2 + \mu^2$$

$$= E[X^2] - \mu^2$$

simple example

→ "heads you win \$1, tails you lose \$1"

$$E[X] = -1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = 0$$

$$Var[X] = ((-1)^2 + 1^2) \cdot \frac{1}{2} - 0^2$$

$$= 2 \cdot \frac{1}{2} - 0$$

$$= 1 - 0$$

$$= 1$$

But if bet were \$10 the variance would be

$$Var[X] = 10^2 - 0^2$$

$$= 100$$

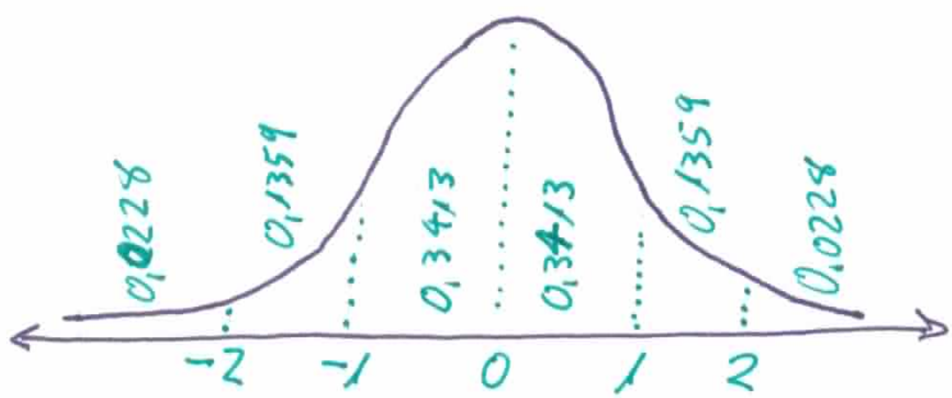
Continuous random variables

→ unlike a coin flip or roll of dice there are an infinite number of possible values, each with probability zero

→ probability density function

- curve above horizontal axis
- with area of one
- probabilities correspond to area under curve

→ standard normal



68% of the area under the curve lies between -1 and +1

$$P[-1 < z < 1] = 0{,}6826$$