Robinson Crusoe Model

The purpose of these notes is to combine the trade models from my *Lecture Notes* with the discussion of consumer preference in the Pindyck and Rubinfeld textbook to form a "Robinson Crusoe model" of relative supply and relative demand. In particular, I hope these notes provide you with a single, unified framework for understanding the material discussed in:

- Pindyck and Rubinfeld chap. 2 and chap. 3 (PPT)
- my Lecture 2 and Lecture 5 (PDF)

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In Daniel Defoe's novel *Robinson Crusoe*, the protagonist (Robinson) is shipwrecked on a deserted island. Even though he no longer interacts with other people, he still has supply and demand decisions to make. On the supply side, he must decide how much beer and how much pizza to produce for himself. On the demand side, he must decide how much beer and how much pizza to consume.

So our first task will be to derive Robinson's relative supply relationship from his production possibilities frontier. Then we will derive Robinson's relative demand relationship from his indifference curve. The point at which the two curves cross determines the equilibrium relative price of pizza and the equilibrium relative quantity of pizza. Assuming that Robinson is producing at a point along his production possibilities frontier, this equilibrium also represents Robinson's utility maximizing combination of pizza and beer.

The second task of these notes is to extend the model to international trade by including Robinson's companion "Friday." In that section, we will show how Robinson and Friday can exploit differences in their labor productivities to reach a higher level of utiltity. Specifically, Robinson and Friday will gain from trade by (partially) specializing in the production of the good in which they have a comparative advantage.

After producing their goods, Robinson and Friday trade beer for pizza until they reach an indifference curve that is higher than any that lies along their production possibility frontiers. The higher indifference curve represents a higher level of utility. Consequently, both Robinson and Friday gain from the exchange.

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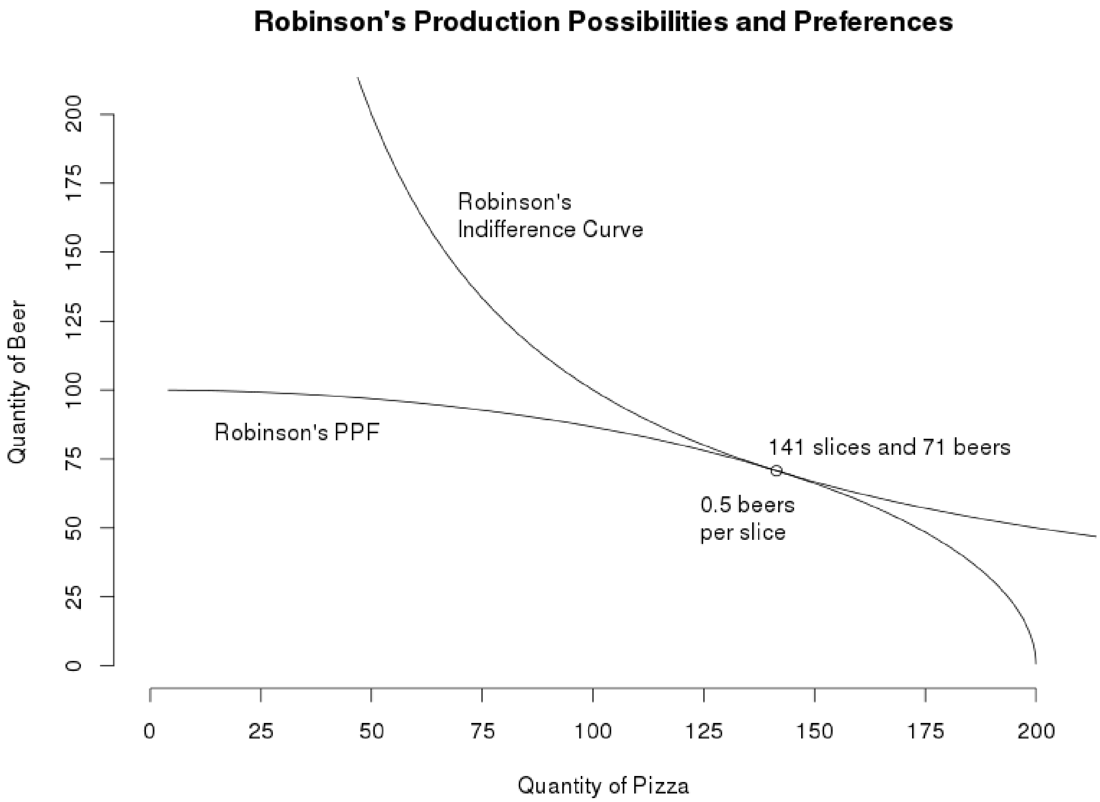
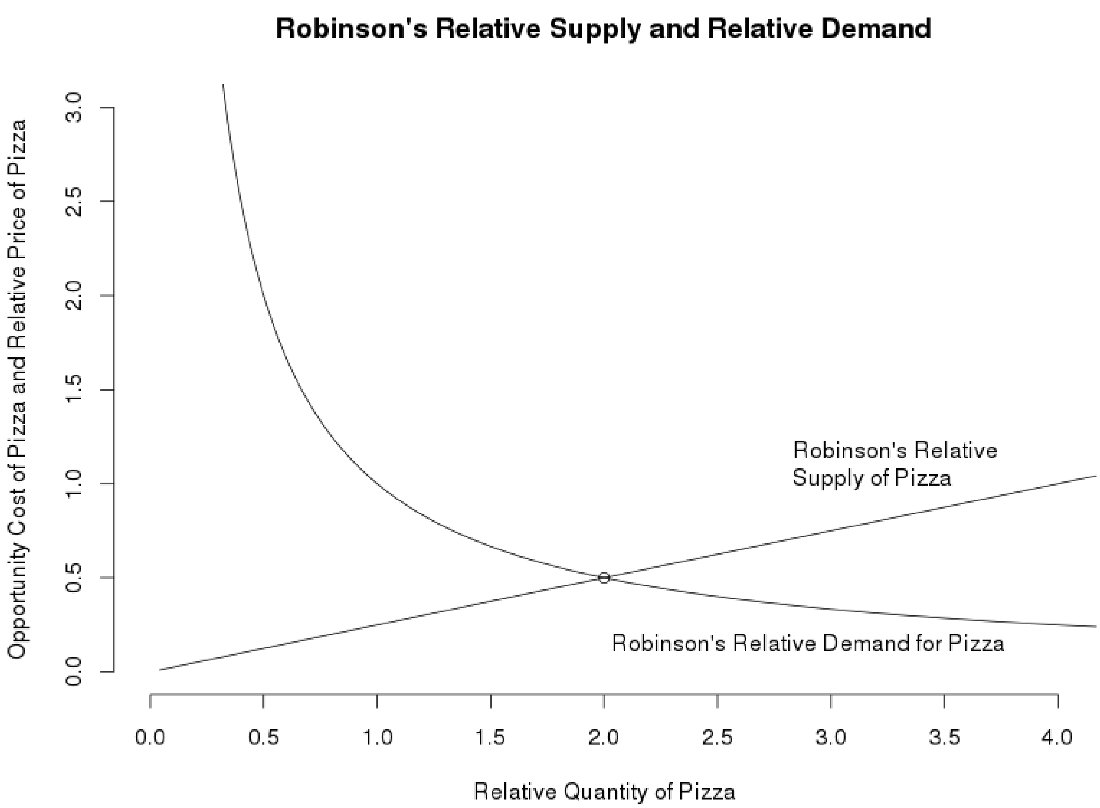
The relative supply relationship gives us the ratio of pizza slices per beer that Robinson is willing to produce for a given relative price of pizza. This relationship is upward sloping because the opportunity cost of producing pizza increases as Robinson produces more pizza (and less beer) due to the effects of diminishing marginal returns. Consequently, an increasingly higher relative price of pizza must be offered to offset Robinson's increasingly higher opportunity cost of producing more pizza and less beer.

Similarly, the relative demand relationship gives us the ratio of pizza slices per beer that Robinson is willing to consume for a given relative price of pizza. The downward slope of the relative demand relationship reflects the downward slope of the indifference curve. Specifically, a given indifference curve represents a particular level of utility. That level of utility consists of a series of combinations of pizza and beer that Robinson would be equally satisfied with. (Not more satisfied. Not less satisfied. Equally satisfied). This is why we say that: "Robinson's utility is held constant along that indifference curve."

So if we were to increase Robinson's consumption of pizza while keeping him at the same level of utility, we would have to reduce his consumption of beer. Assuming that Robinson's utility function is homothetic and assuming that it exhibits diminishing marginal returns, the number of beers that Robinson is willing to forego – to stay on the same indifference curve – becomes smaller as his total consumption of pizza rises. Consequently, inducing Robinson to consume increasingly more pizza and increasingly less beer, requires an increasingly lower relative price of pizza. The relative demand curve represents this relationship.

Robinson's objective is to maximize his utility, but he faces a constraint. Ideally, he would like to consume an infinite amount of pizza and beer. In practice however, he cannot consume more than he produces, so Robinson's own


Robinson's Production Possibilities and Preferences
Robinson's
Indifference Curve
Robinson's PPF
141 slices and 71 beers
0.5 beers
per slice
Quantity of Beer
0
25
50
75
100
125
150
175
200
0
25
50
75
100
125
150
175
200
Quantity of Pizza


Robinson's Relative Supply and Relative Demand
Robinson's Relative
Supply of Pizza
Robinson's Relative Demand for Pizza
Opportunity Cost of Pizza and Relative Price of Pizza
0.0
0.5
1.0
1.5
2.0
2.5
3.0
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
Relative Quantity of Pizza

production possibilities frontier constrains the level of utility that he can attain. To maximize his utility subject to this constraint, Robinson produces at a point along his production possibilities frontier that is tangent to the highest possible indifference curve that he can reach (given his production possibilities).

Because the production possibilities frontier and the indifference curve are tangent to each other at that point, their slopes are equal. In the example on the previous page, their slopes are equal to half a beer per slice at the point of tangency.

The slope of the Robinson's production possibilities frontier represents Robinson's opportunity cost of producing pizza and (therefore) forms one part of the relative supply relationship. The other part of the relative supply relationship is the relative quantity of pizza that Robinson produces. In the example on the previous page, when the relative price of pizza is equal to half a beer per slice, Robinson produces 141 slices and 71 beers. Thus, the relative quantity of pizza that he supplies is two pizza slices per beer.

The slope of the Robinson's indifference curve represents Robinson's marginal rate of substitution of pizza for beer. To maximize his utility, Robinson sets this ratio equal to the relative price of pizza. Consequently, it forms one part of the relative demand relationship. The other part of the relative demand relationship is the relative quantity of pizza that Robinson consumes. In the example on the previous page, when the relative price of pizza is equal to half a beer per slice, Robinson consumes 141 slices and 71 beers. Thus, the relative quantity of pizza that he demands is two pizza slices per beer.

Because Robinson's consumption of pizza and beer is equal to his production of pizza and beer at the relative price of half a beer per pizza slice, equilibrium occurs at that relative price. So we say that the "equilibrium relative price of pizza" is half a beer per pizza slice, while the "equilibrium relative quantity of pizza" is two pizza slices per beer (i.e. 141 slices and 71 beers).

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To provide a mathematical example, let's assume that Robinson has $L_{RC} = 10{,}000$ units of labor that he can allocate to the production of pizza or the production of beer. How much pizza and beer he produces depends on the amount of labor that Robinson allocates to the production of each. We'll assume that Robinson's production of pizza and beer are both subject to diminishing marginal returns and take the following form.

The quantity of pizza that Robinson produces is equal to twice the square root of the labor he allocates to the production of pizza:

$$pizza_{RC} = 2 \cdot \sqrt{L_{RC,pizza}}$$

The quantity of beer that Robinson produces is equal to the square root of the labor he allocates to the production of beer:

$$beer_{RC} = \sqrt{L_{RC,beer}}$$

But because Robinson only has 10,000 units of labor to allocate between pizza and beer, the quantity of beer that he produces declines as he produces more pizza. In other words, Robinson's labor constraint implies that:

$$10,000 = L_{RC,beer} + L_{RC,pizza} \tag{1}$$

$$L_{RC,beer} = 10,000 - L_{RC,pizza}$$

Because the amount of pizza that Robinson produces depends on the amount of labor that he allocates to the production of pizza, the amount of labor that Robinson allocates to the production of beer falls as the quantity of pizza that he produces rises:

$$L_{RC,beer} = 10,000 - \left(\frac{pizza_{RC}}{2}\right)^2$$

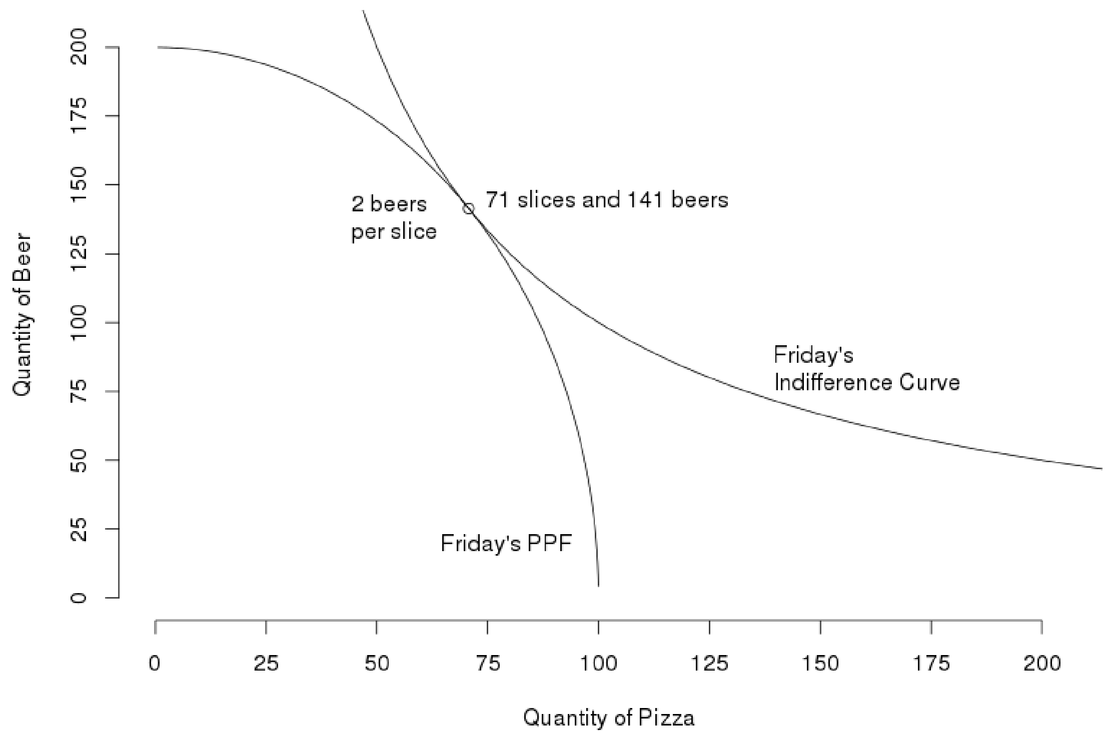
Therefore, we can express the quantity of beer that Robinson produces as a function of the quantity of pizza that he produces:

$$beer_{RC} = \sqrt{10,000 - \left(\frac{pizza_{RC}}{2}\right)^2} \tag{2}$$

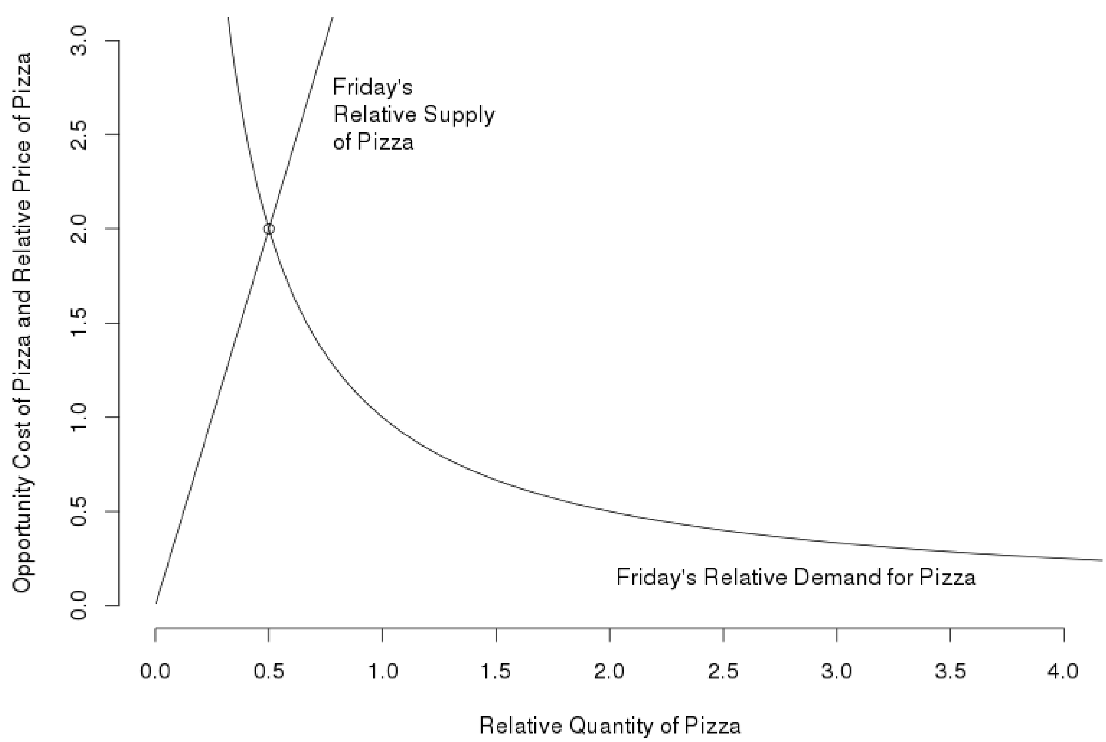
Equation 2 represents Robinson's production possibilities frontier. The slope of that equation represents Robinson's opportunity cost of producing pizza:

$$\left.\frac{d\ beer_{RC}}{d\ pizza_{RC}}\right|_{L_{RC}=10,000} = -\frac{1}{4} \cdot \frac{pizza_{RC}}{beer_{RC}} \tag{3}$$

Friday's Production Possibilities and Preferences



Friday's Relative Supply and Relative Demand



Equation 3 shows two things. First, it shows that Robinson's opportunity cost of producing pizza an increasing function of the relative quantity of pizza that he produces. Second, because equation 3 expresses the the relationship between Robinson's opportunity cost of producing pizza and the relative quantity of pizza that he produces, it represents Robinson's relative supply curve.

Just as we derived Robinson's relative supply curve from his production possibilities, we will now derive Robinson's relative demand curve from his preferences. Specifically, we'll assume that Robinson's utility function takes the following form:

$$U_{RC} = \sqrt{pizza_{RC} \cdot beer_{RC}} \tag{4}$$

We can obtain an expression for Robinson's indifference curve by rearranging the terms in equation 4 and holding utility constant:

$$beer_{RC} = \frac{\overline{U_{RC}}^2}{pizza_{RC}} \tag{5}$$

To derive Robinson's relative demand curve, we next note that use of a homothetic utility function (like the Cobb-Douglas utility function in equation 4) implies that the ratio of pizza-to-beer that Robinson demands will *only* change when the relative price of pizza changes (i.e. it will not change in response to a change in purchasing power that pushes Robinson to another indifference curve).

At a utility maximum, the slope of the indifference curve (i.e. "marginal rate of substitution of pizza for beer") is equal to the relative price of pizza. Consequently, we can obtain Robinson's relative demand for pizza from the slope of his indifference curve:

$$\left.\frac{d\, beer_{RC}}{d\, pizza_{RC}}\right|_{U=\overline{U}} = -\frac{1}{(pizza_{RC}/beer_{RC})} \tag{6}$$

The intersection of Robinson's relative supply and relative demand curves determines Robinson's equilibrium relative price of pizza (half a beer per slice) and equilibrium relative quantity of pizza (two slices per beer = $\frac{141\ slices_{RC}}{71\ beers_{RC}}$).

$$\begin{aligned} slope\ IC &= slope\ PPF \\ \left.\frac{d\, beer_{RC}}{d\, pizza_{RC}}\right|_{U=\overline{U}} &= \left.\frac{d\, beer_{RC}}{d\, pizza_{RC}}\right|_{L_{RC}=10{,}000} \\ -\frac{1}{(pizza_{RC}/beer_{RC})} &= -\frac{1}{4} \cdot \frac{pizza_{RC}}{beer_{RC}} \end{aligned} \tag{7}$$

which implies that in equilibrium:

$$\begin{aligned} 4 &= \left(\frac{pizza_{RC}}{beer_{RC}}\right)^2 \\ 2 \cdot beer_{RC} &= pizza_{RC} \end{aligned}$$

the number of pizza slices that Robinson produces and consumes is two times the number of beers that he produces and consumes. That equilibrium relative quantity of pizza corresponds to an equilibrium relative price of half a beer per slice. It also corresponds to production and consumption of 141 slices and 71 beers.

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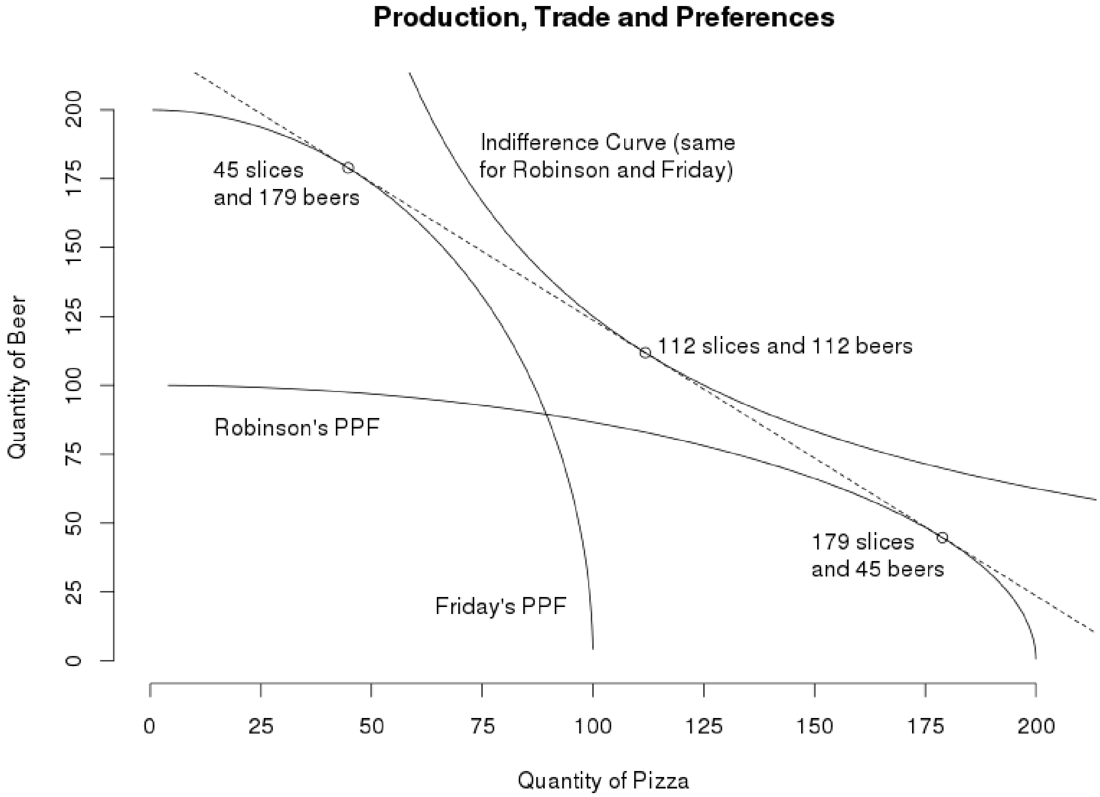
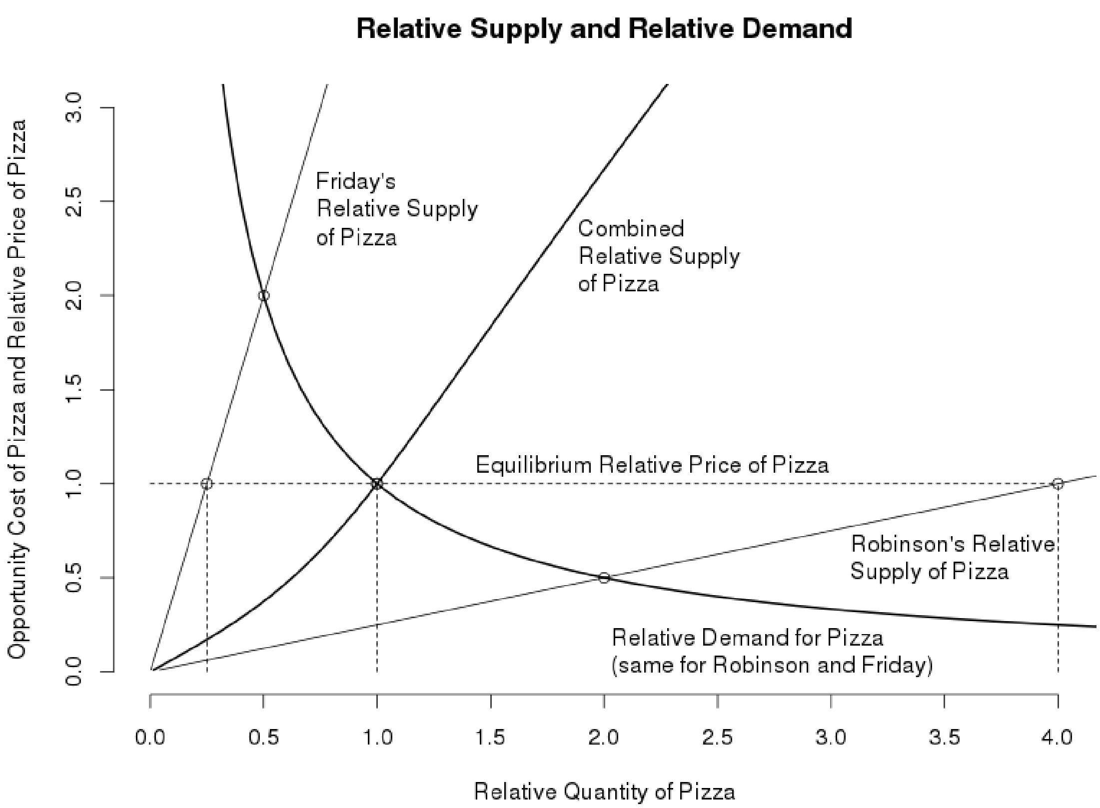
The second task of these notes is to show how Robinson's decisions change when he is offered a new set of opportunities. Specifically, we'll alter the relative price of pizza that Robinson faces by giving him a companion named "Friday" who can produce twice as much beer as Robinson can for the same amount of labor. Robinson, on the other hand, can produce twice as much pizza as Friday for the same amount of labor. By dividing production along the lines of comparative advantage, Robinson and Friday can trade for mutual gain and attain a higher level of utility.

To hold all other factors constant, we'll assume that Friday and Robinson both have the same utility function (which implies that they both have the same indifference curve and the same relative demand curve).


Production, Trade and Preferences
Indifference Curve (same for Robinson and Friday)
45 slices and 179 beers
112 slices and 112 beers
Robinson's PPF
179 slices and 45 beers
Friday's PPF
Quantity of Beer
Quantity of Pizza
0
25
50
75
100
125
150
175
200


Relative Supply and Relative Demand
Friday's Relative Supply of Pizza
Combined Relative Supply of Pizza
Equilibrium Relative Price of Pizza
Robinson's Relative Supply of Pizza
Relative Demand for Pizza (same for Robinson and Friday)
Opportunity Cost of Pizza and Relative Price of Pizza
Relative Quantity of Pizza
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0

Because their labor productivities differ, it makes intuitive sense for Robinson and Friday to (partially) specialize in what they're relatively more efficient at producing and then make mutually beneficial trades. In the example below, Robinson will supply relatively more pizza (and less beer) than he'll consume when he has the opportunity to trade some of his pizzas for some of Friday's beers. Conversely, Friday will supply relatively less pizza (and more beer) than he'll consume when he has the opportunity to trade with Robinson.

Our assumption of different labor productivities implies that Robinson and Friday have different relative supply functions, so we need to derive Friday's relative supply function. We'll assume that Friday has $L_{Fri} = 10{,}000$ units of labor that to allocate to the production of pizza or beer (just like Robinson), but we'll assume that Friday is twice as productive than Robinson in the production of beer, while Robinson is twice as productive as Friday in the production of pizza.

The quantity of pizza that Friday produces is equal to the square root of the labor he allocates to the production of pizza:

$$pizza_{Fri} = \sqrt{L_{Fri,pizza}}$$

The quantity of beer that Friday produces is equal to twice the square root of the labor he allocates to the production of beer:

$$beer_{Fri} = 2 \cdot \sqrt{L_{Fri,beer}}$$

Using the same mathematics that we developed for Robinson, we can express the quantity of beer that Friday produces as a function of the quantity of pizza that he produces:

$$beer_{Fri} = 2 \cdot \sqrt{10,000 - (pizza_{Fri})^2} \tag{8}$$

which represents Friday's production possibilities frontier. Its slope represents his opportunity cost of producing pizza:

$$\left.\frac{d\,beer_{Fri}}{d\,pizza_{Fri}}\right|_{L_{Fri}=10,000} = -4 \cdot \frac{pizza_{Fri}}{beer_{Fri}} \tag{9}$$

As before, equation 9 represents Friday's relative supply curve because it expresses the the relationship between Friday's opportunity cost of producing pizza and the relative quantity of pizza that he produces.

Comparing the two, one can see that Friday's relative supply curve has a steeper slope than Robinson's. This difference reflects Robinson's comparative advantage in the production of pizza and Friday's comparative advantage in the production of beer.

Betweeen the Robinson and Friday's relative supply curves lies the combined relative supply curve which represents the ratio of total pizza to total beer produced at each relative price of pizza (where "total" refers to Robinson and Friday's combined production).

The intersection of the combined relative supply curve and the relative demand curve determines the equilibrium relative price and equilibrium relative quantity in the combined market. In the combined market, the equilibrium relative price will be one beer per slice and the equilibrium relative quantity will be one slice per beer = $\frac{224\ slices}{224\ beers}$.

Because the new equilibrium relative price exceeds the one that Robinson faced when he did not trade with Friday, Robinson will produce more pizza (and less beer) than he did before the opportunity to trade with Friday arose.

Specifically, the new equilibrium relative price of one beer per slice pizza intersects Robinson's relative supply curve at a relative quantity of four slices per beer, so Robinson will produce 179 slices and 45 beers. Notice however that the new equilibrium relative price of pizza intersects Robinson's relative demand curve at one slice per beer, so Robinson will trade 67 of the pizza slices that he produced for 67 of the beers that Friday produced to obtain a (utility-maximizing) final basket of 112 slices and 112 beers.

Whereas Robinson supplies relatively more pizza (and less beer) than he consumes when the relative price is one beer per slice, Friday supplies relatively less pizza (and more beer). Specifically, Friday will produce 45 slices and 179 beers, then trade 67 of the beers that he produced for 67 of the pizza slices that Robinson produced and thus consume the same (utility-maximizing) final basket of 112 slices and 112 beers.

The opportunity to specialize and trade generates gains for both Robinson and Friday. By (partially) specializing in the production of the good in which they each have a comparative advantage, both Robinson and Friday can reach a higher level of utility than they could if they did not trade.

The higher level of utility that Robinson and Friday reach is depicted by the indifference curve that lies along the trade line (with a slope of $-1\frac{beer}{slice}$) that connects their production possibility frontiers. Because this indifference curve lies above both of their production possibility frontiers, we can quickly see that both Robinson and Friday reach a level of utility that they could not have reached in the absence of trade.

Another way to show the gain from trade is to insert the quantities of pizza and beer that they produce into their utility functions and compare the difference before and after trade.

In Robinson's case, he consumed 141 slices and 71 beers before trade, for total utility of $100 = \sqrt{141 \cdot 71}$. After trade, he consumes 112 slices and 112 beers, which gives him a new (higher) total utility of $112 = \sqrt{112 \cdot 112}$.

In Friday's case, he consumed 71 slices and 141 beers before trade, for total utility of $100 = \sqrt{71 \cdot 141}$. After trade, he consumes 112 slices and 112 beers, which gives him a new (higher) total utility of $112 = \sqrt{112 \cdot 112}$.