ECONOMETRICS PROBLEM SET

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PROBLEM #3 - DERIVE THE MLE ESTIMATES of α, β and σ^2

$$\varepsilon_i = y_i - \alpha - \beta x_i$$

$$\varepsilon \sim N(0, \sigma^2)$$

Likelihood Function $$\mathcal{L}(\alpha, \beta, \sigma^2) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{1}{2}\left(\frac{y_i - \alpha - \beta x_i}{\sigma^2}\right)^2}$$

Log Likelihood Function $$\ln \mathcal{L}(\alpha, \beta, \sigma^2) = \frac{-N}{2} \ln(2\pi) - \frac{N}{2} \ln(\sigma^2) - \frac{1}{2} \sum \frac{(y - \alpha - \beta x)^2}{\sigma^2}$$

1. first order conditions

$$\boxed{\frac{\partial \ln \mathcal{L}}{\partial \alpha} = \frac{1}{\sigma^2} \sum (y - \alpha - \beta x) = 0}$$

$$\boxed{\frac{\partial \ln \mathcal{L}}{\partial \beta} = \frac{1}{\sigma^2} \sum (y - \alpha - \beta x) x = 0}$$

define $\gamma \equiv \sigma^2$ and differentiate with respect to γ

$$\boxed{\frac{\partial \ln \mathcal{L}}{\partial \gamma} = \frac{-N}{2\gamma} + \frac{1}{2\gamma^2} \sum (y - \alpha - \beta x)^2 = 0}$$

2. define $\bar{x} \equiv \frac{1}{N}\Sigma x$ and $\bar{y} \equiv \frac{1}{N}\Sigma y$

$$\frac{\partial \ln \mathcal{L}}{\partial \alpha} = \frac{1}{\gamma}\Sigma(y-\alpha-\beta x) = 0$$

implies that:

$$\frac{1}{N}\Sigma(y-\alpha-\beta x) = 0$$

$$\frac{1}{N}\Sigma y - \alpha - \beta\cdot\frac{1}{N}\Sigma x = 0$$

$$\bar{y} - \alpha - \beta\bar{x} = 0$$

$$\boxed{\alpha = \bar{y} - \beta\bar{x}}$$

$$\frac{\partial \ln \mathcal{L}}{\partial \beta} = \frac{1}{\gamma}\Sigma(y-\alpha-\beta x)\,x = 0$$

implies that:

$$\frac{1}{N}\Sigma\left[(y-\alpha-\beta x)\,x\right] = 0$$

inserting $\alpha = \bar{y} - \beta x$ and rearranging terms:

$$\frac{1}{N}\Sigma\left[\left(y-\bar{y}+\beta(x-\bar{x})\right)x\right] = 0$$

$$\frac{1}{N}\Sigma(y-\bar{y})\,x = \beta\cdot\frac{1}{N}\Sigma(x-\bar{x})\,x$$

solving for β:

$$\boxed{\beta = \frac{\frac{1}{N}\Sigma(y-\bar{y})x}{\frac{1}{N}\Sigma(x-\bar{x})x} = \boxed{\frac{\frac{1}{N}\Sigma(x-\bar{x})(y-\bar{y})}{\frac{1}{N}\Sigma(x-\bar{x})} = \frac{cov(x,y)}{Var(x)}}}$$

for details see answers to questions 3 + 4 of problem #1

2. (continued)

$$\frac{\partial \ln \mathcal{L}}{\partial \gamma} = \frac{-N}{2\gamma} + \frac{1}{2\gamma^2}\Sigma(y-\alpha-\beta x)^2 = 0$$

implies that:

$$\frac{1}{\gamma}\Sigma(y-\alpha-\beta x)^2 = N$$

$$\boxed{\gamma = \frac{1}{N}\Sigma(y-\alpha-\beta x)^2}$$

3. second-order condition + Hessian matrix

first partials $\frac{\partial \ln \mathcal{L}}{\partial \alpha} = \frac{1}{\gamma}\Sigma(y-\alpha-\beta x)$ $\quad \frac{\partial \ln \mathcal{L}}{\partial \beta} = \frac{1}{\gamma}\Sigma(y-\alpha-\beta x)x$ $\quad \frac{\partial \ln \mathcal{L}}{\partial \gamma} = \frac{-N}{2\gamma} + \frac{1}{2\gamma^2}\Sigma(y-\alpha-\beta x)^2$

second "own" partials $\boxed{\frac{\partial^2 \ln \mathcal{L}}{\partial \alpha^2} = \frac{-N}{\gamma} < 0}$ $\quad \boxed{\frac{\partial^2 \ln \mathcal{L}}{\partial \beta^2} = \frac{-1}{\gamma}\Sigma x^2 < 0}$ $\quad \frac{\partial^2 \ln \mathcal{L}}{\partial \gamma^2} = \frac{N}{2\gamma^2} - \frac{1}{\gamma^3}\Sigma(y-\alpha-\beta x)^2$

$$= \frac{N}{\gamma^2}\left[\frac{1}{2} - \frac{1}{\gamma}\cdot\frac{\Sigma(y-\alpha-\beta x)^2}{N}\right]$$

at a maximum: $\gamma = \frac{1}{N}\Sigma(y-\alpha-\beta x)^2$

$$\boxed{\frac{\partial^2 \ln \mathcal{L}}{\partial \gamma^2} = \frac{-N}{2\gamma^2} < 0}$$

"cross partials"

$$\frac{\partial^2 \ln \mathcal{L}}{\partial \alpha\,\partial \beta} = \frac{-1}{\gamma}\cdot\Sigma x = \frac{-N\bar{x}}{\gamma}$$

$$\frac{\partial^2 \ln \mathcal{L}}{\partial \alpha\,\partial \gamma} = \frac{-1}{\gamma^2}\Sigma(y-\alpha-\beta x) \leftarrow \underline{\underline{\text{zero}}} \text{ by 1st OC}$$

$$\frac{\partial^2 \ln \mathcal{L}}{\partial \beta\,\partial \gamma} = \frac{-1}{\gamma^2}\Sigma(y-\alpha-\beta x)x \leftarrow \underline{\underline{\text{zero}}} \text{ by 1st OC}$$

3. (continued)

Hessian Matrix and determinant

$$H = \begin{bmatrix} \frac{\partial^2 \ln \mathcal{L}}{\partial \alpha^2} & \frac{\partial^2 \ln \mathcal{L}}{\partial \alpha \partial \beta} & \frac{\partial^2 \ln \mathcal{L}}{\partial \alpha \partial \gamma} \\ \frac{\partial^2 \ln \mathcal{L}}{\partial \alpha \partial \beta} & \frac{\partial^2 \ln \mathcal{L}}{\partial \beta^2} & \frac{\partial^2 \ln \mathcal{L}}{\partial \beta \partial \gamma} \\ \frac{\partial^2 \ln \mathcal{L}}{\partial \alpha \partial \gamma} & \frac{\partial^2 \ln \mathcal{L}}{\partial \beta \partial \gamma} & \frac{\partial^2 \ln \mathcal{L}}{\partial \gamma^2} \end{bmatrix} = \begin{bmatrix} \frac{-N}{\gamma} & \frac{-N\bar{x}}{\gamma} & 0 \\ \frac{-N\bar{x}}{\gamma} & \frac{-N}{\gamma} \cdot \frac{\Sigma x^2}{N} & 0 \\ 0 & 0 & \frac{-N}{\gamma} \cdot \frac{1}{2\gamma} \end{bmatrix}$$

$$H = \frac{-N}{\gamma} \cdot \begin{bmatrix} 1 & \bar{x} & 0 \\ \bar{x} & \frac{1}{N}\Sigma x^2 & 0 \\ 0 & 0 & \frac{1}{2\gamma} \end{bmatrix}$$

determinant of the Hessian matrix

$$|H| = \left(\frac{-N}{\gamma}\right)^3 \cdot \left[\frac{1}{2\gamma} \cdot \left(\frac{1}{N}\Sigma x^2 - \bar{x}^2\right)\right]$$

$$\boxed{|H| = \frac{-N^3}{2\gamma^4} \cdot \mathrm{var}(x) < 0}$$

4. information matrix

$$I = -1 * H^{-1}$$

the Hessian is relatively easy to invert because there is a block of zeros in the upper right and a block of zeros in the lower left. Consequently, we only have to invert the 2x2 in the upper left and the "1x1" in the lower right

invert

$$H = \begin{bmatrix} -\frac{N}{\gamma} * \begin{bmatrix} 1 & \bar{X} \\ \bar{X} & \frac{1}{N}\Sigma x^2 \end{bmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & \quad 0 \end{matrix} & -\frac{N}{\gamma} \cdot \frac{1}{2\gamma} \end{bmatrix}$$

invert

$$-1 * H^{-1} = \begin{bmatrix} \frac{\gamma}{N} * \frac{1}{var(x)} * \begin{bmatrix} \frac{1}{N}\Sigma x^2 & -\bar{x} \\ -\bar{x} & 1 \end{bmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & \quad 0 \end{matrix} & \frac{2\gamma^2}{N} \end{bmatrix}$$

4. (continued)

information matrix

$$-1 * H^{-1} = \frac{\gamma}{N} * \begin{bmatrix} \frac{\frac{1}{N}\Sigma x^2}{var(x)} & \frac{-\bar{x}}{var(x)} & 0 \\ \frac{-\bar{x}}{var(x)} & \frac{1}{var(x)} & 0 \\ 0 & 0 & 2\gamma \end{bmatrix}$$

The square roots of the diagonal elements of the information matrix are the standard errors of our estimates.

Recalling that $\gamma = \sigma^2$ (and therefore: $\sqrt{\gamma} = \sigma$),

$$\text{std. error of } \hat{\alpha} = \hat{\sigma} \cdot \sqrt{\frac{\frac{1}{N}\Sigma x^2}{N \cdot var(x)}} = \sigma \cdot \sqrt{\frac{\frac{1}{N}\Sigma x^2}{\Sigma(x-\bar{x})^2}}$$

$$\text{std error of } \hat{\beta} = \hat{\sigma} \cdot \sqrt{\frac{1}{N \cdot var(x)}} = \sigma \cdot \sqrt{\frac{1}{\Sigma(x-\bar{x})^2}}$$

$$\text{std error of } \hat{\sigma}^2 = \hat{\sigma}^2 \cdot \sqrt{\frac{2}{N}}$$

e↗

5. When the log likelihood surface comes to a sharp peak along a given dimension, the second derivative of the log likelihood function ~~will be~~ (second "own partial") will be relatively large.

The standard errors are ~~also~~ obtained from an inverted matrix of second partial derivatives.

Therefore when the second "own partials" are large, the standard error is small.

So when the log likelihood function comes to a sharp peak we ~~do~~ have a better estimate of the parameter and a correspondingly smaller standard error.