ECONOMETRICS PROBLEM SET

Eryk Wdowiak

PROBLEM #2 - DERIVE THE MLE ESTIMATES of μ and σ^2

Likelihood Function $$\mathcal{L}(\mu,\sigma^2) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{1}{2}\cdot\frac{(x-\mu)^2}{\sigma^2}}$$

1. To obtain the Log Likelihood Function, recall that the log of a product is equal to the sum of the logs

$$\ln(a \cdot b) = \ln(a) + \ln(b)$$

and recall that the log of an exponential function is the exponent

$$\ln(e^b) = b$$

Therefore, the Log Likelihood Function is:

$$\ln \mathcal{L}(\mu,\sigma^2) = \sum\left(-\frac{1}{2}\cdot\ln(2\pi\sigma^2) - \frac{1}{2}\cdot\frac{(x-\mu)^2}{\sigma^2}\right)$$

We will differentiate with respect to σ^2, so define $\boxed{\gamma \equiv \sigma^2}$ and rearrange terms

$$\ln \mathcal{L}(\mu,\gamma) = -\frac{1}{2}\cdot\sum\left[\ln(2\pi) + \ln(\gamma) + \frac{(x-\mu)^2}{\gamma}\right]$$

1. (continued)

Rearrange terms one more time

Log Likelihood Function

$$\boxed{\ln \mathcal{L}(\mu,\gamma) = \frac{-N}{2}\ln(2\pi) - \frac{N}{2}\ln(\gamma) - \frac{1}{2}\cdot\sum\frac{(x-\mu)^2}{\gamma}}$$

2. first-order conditions

$$\frac{\partial \ln \mathcal{L}}{\partial \mu} = \frac{-1}{2\gamma}\cdot\sum\frac{\partial(x-\mu)^2}{\partial(x-\mu)}\cdot\frac{\partial(x-\mu)}{\partial\mu} = 0$$

$$= \frac{-1}{2\gamma}\cdot\sum\left[2\cdot(x-\mu)\cdot(-1)\right] = 0$$

$$\boxed{\frac{\partial \ln \mathcal{L}}{\partial \mu} = \frac{1}{\gamma}\sum(x-\mu) = 0}$$

$$\frac{\partial \ln \mathcal{L}}{\partial \gamma} = -\frac{N}{2}\cdot\frac{\partial \ln(\gamma)}{\partial\gamma} - \frac{1}{2}\cdot\left[\sum(x-\mu)^2\right]\cdot\frac{\partial(1/\gamma)}{\partial\gamma} = 0$$

$$= \frac{-N}{2}\cdot\frac{1}{\gamma} - \frac{\sum(x-\mu)^2}{2}\cdot\frac{1}{\gamma^2} = 0$$

$$\boxed{\frac{\partial \ln \mathcal{L}}{\partial \gamma} = \frac{-1}{2\gamma}\left(N - \frac{\sum(x-\mu)^2}{\gamma}\right) = 0}$$

3. obtain the estimate of μ

$$\frac{\partial \ln \mathcal{L}}{\partial \mu} = \frac{1}{\gamma} \Sigma (x - \mu) = 0$$

implies that at a maximum:

$$\Sigma (x - \mu) = 0$$

$$\Sigma x - N \cdot \mu = 0$$

rearranging terms:

$$\boxed{\mu = \frac{1}{N} \Sigma x}$$

MLE estimate of the mean

4. obtain the estimate of $\gamma \equiv \sigma^2$

$$\frac{\partial \ln \mathcal{L}}{\partial \gamma} = \frac{-1}{2\gamma} \left(N - \frac{\Sigma (x-\mu)^2}{\gamma} \right) = 0$$

implies that at a maximum:

$$N - \frac{\Sigma (x-\mu)^2}{\gamma} = 0$$

rearranging terms:

$$\boxed{\gamma = \frac{1}{N} \Sigma (x - \mu)^2}$$

MLE estimate of the variance

5, 6, 7 the second-order conditions for a maximum require that the "own partials" be negative and that the Hessian determinant be positive

first partials $\quad \dfrac{\partial \ln \mathcal{L}}{\partial \mu} = \dfrac{1}{\gamma}\Sigma(x-\mu) \qquad \dfrac{\partial \ln \mathcal{L}}{\partial \gamma} = \dfrac{-N}{2\gamma} + \dfrac{\Sigma(x-\mu)^2}{2\gamma^2}$

second "own" partials $\quad \boxed{\dfrac{\partial^2 \ln \mathcal{L}}{\partial \mu^2} = \dfrac{-N}{\gamma} < 0} \qquad \dfrac{\partial^2 \ln \mathcal{L}}{\partial \gamma^2} = \dfrac{N}{2\gamma^2} - \dfrac{\Sigma(x-\mu)^2}{\gamma^3}$

$$= \frac{N}{\gamma^2}\left(\frac{1}{2} - \frac{1}{\gamma}\cdot\frac{\Sigma(x-\mu)^2}{N}\right)$$

"cross partial" $\quad \boxed{\dfrac{\partial^2 \ln \mathcal{L}}{\partial \mu \partial \gamma} = \dfrac{-1}{\gamma^2}\Sigma(x-\mu)}$

at a maximum: $\gamma = \frac{1}{N}\Sigma(x-\mu)^2)$

$$\frac{\partial^2 \ln \mathcal{L}}{\partial \gamma^2} = \frac{N}{\gamma^2}\left(\frac{1}{2} - 1\right)$$

$$\boxed{\frac{\partial^2 \ln \mathcal{L}}{\partial \gamma^2} = \frac{-N}{2\gamma^2} < 0}$$

8. Hessian matrix and determinant

$$H = \begin{bmatrix} \dfrac{\partial^2 \ln \mathcal{L}}{\partial \mu^2} & \dfrac{\partial^2 \ln \mathcal{L}}{\partial \mu \partial \gamma} \\[2ex] \dfrac{\partial^2 \ln \mathcal{L}}{\partial \mu \partial \gamma} & \dfrac{\partial^2 \ln \mathcal{L}}{\partial \gamma^2} \end{bmatrix} = \begin{bmatrix} \dfrac{-N}{\gamma} & \dfrac{-\Sigma(x-\mu)}{\gamma^2} \\[2ex] \dfrac{-\Sigma(x-\mu)}{\gamma^2} & \dfrac{-N}{2\gamma^2} \end{bmatrix}$$

but at a maximum $\Sigma(x-\mu) = 0$, therefore

8. (continued)

$$H = \begin{bmatrix} \frac{-N}{\gamma} & \frac{-\Sigma(x-\mu)}{\gamma^2} \\ \frac{-\Sigma(x-\mu)}{\gamma^2} & \frac{-N}{2\gamma^2} \end{bmatrix} = \begin{bmatrix} \frac{-N}{\gamma} & 0 \\ 0 & \frac{-N}{2\gamma^2} \end{bmatrix}$$

$$\boxed{|H| = \frac{N^2}{2\gamma^3} > 0}$$

9. information matrix $I = -1 * H^{-1}$

$$I = -1 * H^{-1}$$

$$= \begin{bmatrix} \frac{\gamma}{N} & 0 \\ 0 & \frac{2\gamma^2}{N} \end{bmatrix}$$

square roots of the diagonal elements are the standard errors

$$\text{std error of } \hat{\mu} = \sqrt{\frac{\gamma}{N}} = \frac{\hat{\sigma}}{\sqrt{N}}$$

$$\text{std error of } \hat{\gamma} = \sqrt{\frac{2\gamma^2}{N}} = \hat{\sigma}^2 \cdot \sqrt{\frac{2}{N}}$$