ECONOMETRICS PROBLEM SET

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PROBLEM #1 – DERIVE THE OLS ESTIMATES of α and β

$$y_i = \alpha + \beta x_i + \varepsilon_i$$

sum of squared errors:

$$\sum_{i=1}^{N} \varepsilon_i^2 = \sum_{i=1}^{N} (y_i - \alpha - \beta x_i)^2$$

1. first order conditions

$$\underbrace{\frac{\partial \Sigma \varepsilon_i^2}{\partial \alpha} = \Sigma \frac{\partial \varepsilon_i^2}{\partial \alpha}}_{\text{change in the sum is equal to sum of the changes}} = \underbrace{\Sigma \frac{\partial (y_i - \alpha - \beta x_i)^2}{\partial (y_i - \alpha - \beta x_i)} \cdot \frac{\partial (y_i - \alpha - \beta x_i)}{\partial \alpha}}_{\text{chain rule}} = 0$$

1st O.C. for a minimum

$$\frac{\partial \Sigma \varepsilon_i^2}{\partial \alpha} = \Sigma \frac{\partial (y - \alpha - \beta x)^2}{\partial (y - \alpha - \beta x)} \cdot \frac{\partial (y - \alpha - \beta x)}{\partial \alpha} = 0$$

$$= \Sigma \left[2(y - \alpha - \beta x) \cdot (-1) \right] = 0$$

$$\boxed{\frac{\partial \Sigma \varepsilon^2}{\partial \alpha} = -2 \cdot \Sigma (y - \alpha - \beta x) = 0}$$

1st O.C. for a minimum of $\Sigma \varepsilon^2$ with respect to α

1. (continued)

$$\frac{\partial \Sigma \varepsilon^2}{\partial \beta} = \Sigma \frac{\partial \varepsilon^2}{\partial \beta} = \Sigma \frac{\partial (y-\alpha-\beta x)^2}{\partial (y-\alpha-\beta x)} \cdot \frac{\partial (y-\alpha-\beta x)}{\partial \beta} = 0$$

$$= \Sigma \left[2(y-\alpha-\beta x) \cdot (-x) \right] = 0$$

$$\boxed{\frac{\partial \Sigma \varepsilon^2}{\partial \beta} = -2 \cdot \Sigma \left[(y-\alpha-\beta x) \cdot x \right] = 0}$$

1st O.C. for a minimum of $\Sigma \varepsilon^2$ with respect to β

2. define $\bar{x} \equiv \frac{1}{N} \Sigma x_i$ and $\bar{y} \equiv \frac{1}{N} \Sigma y_i$

$$\frac{\partial \Sigma \varepsilon^2}{\partial \alpha} = -2 \cdot \Sigma (y-\alpha-\beta x) = 0$$

which implies that at a minimum:

$$\frac{1}{N} \Sigma (y-\alpha-\beta x) = 0$$

$$\frac{1}{N} \Sigma y - \alpha - \beta \cdot \frac{1}{N} \Sigma x = 0$$

$$\bar{y} - \alpha - \beta \bar{x} = 0$$

rearranging terms yields OLS estimate of α

$$\boxed{\alpha = \bar{y} - \beta \bar{x}}$$

3.

$$\frac{\partial \Sigma \varepsilon^2}{\partial \beta} = -2 \cdot \Sigma\left[(y-\alpha-\beta x)\, x\right] = 0$$

which implies that:

$$\frac{1}{N} \Sigma\left[(y-\alpha-\beta x)\, x\right] = 0$$

inserting the estimate of α

$$\alpha = \bar{y} - \beta \bar{x}$$

$$\downarrow$$

$$\frac{1}{N} \Sigma\left[(y-[\bar{y}-\beta\bar{x}]-\beta x)\, x\right] = 0$$

$$\frac{1}{N} \Sigma\left[\Big(y-\bar{y}+\beta(\bar{x}-x)\Big)\, x\right] = 0$$

$$\frac{1}{N} \Sigma(y-\bar{y})x = \beta \cdot \frac{1}{N} \Sigma(x-\bar{x})x$$

rearranging terms yield the OLS estimate of β

$$\beta = \frac{\frac{1}{N}\Sigma(y-\bar{y})x}{\frac{1}{N}\Sigma(x-\bar{x})x} = \frac{\frac{1}{N}\Sigma xy - \bar{y}\cdot\frac{1}{N}\Sigma x}{\frac{1}{N}\Sigma x^2 - \bar{x}\cdot\frac{1}{N}\Sigma x}$$

inserting $\bar{x} \equiv \frac{1}{N}\Sigma x$ and $\bar{y} \equiv \frac{1}{N}\Sigma y$

$$\boxed{\beta = \frac{\frac{1}{N}\Sigma xy - \bar{x}\cdot\bar{y}}{\frac{1}{N}\Sigma x^2 - \bar{x}^2}}$$

4. rearrange terms to show that the OLS estimate of β is the ratio of $\text{cov}(x,y)$ to $\text{var}(x)$

$$\beta = \frac{\frac{1}{N}\sum xy - \bar{x}\bar{y}}{\frac{1}{N}x^2 - \bar{x}^2}$$

adding "zero" to numerator + denominator:

$$\beta = \frac{\frac{1}{N}\sum xy - \bar{x}\bar{y} + (\bar{x}\bar{y} - \bar{x}\bar{y})}{\frac{1}{N}\sum x^2 - \bar{x}^2 + (\bar{x}^2 - \bar{x}^2)}$$

recalling the definition $\bar{x} \equiv \frac{1}{N}\sum x$ and $\bar{y} \equiv \frac{1}{N}\sum y$

$$\beta = \frac{\frac{1}{N}\sum xy - \bar{y}\cdot\frac{1}{N}\sum x + \bar{x}\bar{y} - \bar{x}\cdot\frac{1}{N}\sum y}{\frac{1}{N}\sum x^2 - \bar{x}\frac{1}{N}\sum x + \bar{x}^2 - \bar{x}\cdot\frac{1}{N}\sum x}$$

rearranging terms and collecting all terms inside summations:

$$\beta = \frac{\frac{1}{N}\sum(xy - x\bar{y} - \bar{x}y + \bar{x}\bar{y})}{\frac{1}{N}\sum(x^2 - 2\bar{x}\cdot x + \bar{x}^2)}$$

$$\boxed{\beta = \frac{\frac{1}{N}\sum(x-\bar{x})(y-\bar{y})}{\frac{1}{N}\sum(x-\bar{x})^2} = \frac{\text{cov}(x,y)}{\text{var}(x)}}$$

$$\boxed{\begin{array}{l} \underline{\text{OLS}} \quad \underline{\text{ESTIMATES}} \\ \hat{\alpha} = \bar{y} - \hat{\beta}\bar{x} \\ \hat{\beta} = \dfrac{\frac{1}{n}\Sigma(x-\bar{x})(y-\bar{y})}{\frac{1}{n}\Sigma(x-\bar{x})^2} = \dfrac{\mathrm{cov}(x,y)}{\mathrm{var}(x)} \end{array}}$$

5, 6, 7 the second-order conditions for a minimum require that the "own partials" be positive and that the Hessian determinant be positive

first partials
$$\frac{\partial \Sigma \varepsilon^2}{\partial \alpha} = -2\cdot\Sigma(y-\alpha-\beta x) \qquad \frac{\partial \Sigma \varepsilon^2}{\partial \beta} = -2\cdot\Sigma(y-\alpha-\beta x)x$$

second "own" partials
$$\boxed{\frac{\partial^2 \Sigma \varepsilon^2}{\partial \alpha^2} = 2\cdot N > 0} \qquad \boxed{\frac{\partial \Sigma \varepsilon^2}{\partial \beta^2} = 2\cdot\Sigma x^2 > 0}$$

"cross partial"
$$\boxed{\frac{\partial^2 \Sigma \varepsilon^2}{\partial \alpha\, \partial \beta} = 2\cdot\Sigma x}$$

8. Hessian matrix and determinant

$$H = \begin{bmatrix} \frac{\partial \Sigma \varepsilon^2}{\partial \alpha^2} & \frac{\partial \Sigma \varepsilon^2}{\partial \beta^2} \\ \frac{\partial \Sigma \varepsilon^2}{\partial \alpha \partial \beta} & \frac{\partial \Sigma \varepsilon^2}{\partial \alpha \partial \beta} \end{bmatrix} = \begin{bmatrix} 2N & 2\Sigma x \\ 2\Sigma x & 2\Sigma x^2 \end{bmatrix} = 2\cdot\begin{bmatrix} N & \Sigma x \\ \Sigma x & \Sigma x^2 \end{bmatrix}$$

$$|H| = 4\left(N\cdot\Sigma x^2 - (\Sigma x)^2\right)$$

8. (continued)

To show that the Hessian determinant is positive, we'll show that it is a function of the variance of x (which, of course, must be positive because it is a sum of squares)

$$|H| = 4\cdot\left(N\cdot\Sigma x^2 - (\Sigma x)^2\right)$$

$$= 4\cdot N^2\cdot\left(\frac{\Sigma x^2}{N} - \left(\frac{\Sigma x}{N}\right)^2\right)$$

$$= 4N^2\left(\frac{1}{N}\Sigma x^2 - \bar{x}^2\right)$$

$$= 4N^2\cdot\left(\frac{1}{N}\Sigma(x-\bar{x})^2\right)$$

$$\boxed{|H| = 4N^2\cdot \text{var}(x) > 0}$$