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Econometrics
shape of the likelihood function

First, load the "distrib" package.

```
(%i1) load(distrib)$
```

Now, randomly draw 10 values from the standard normal.

y = alpha + beta*x + u

```
(%i5) N:10$
x:random_normal(0,1,N)$
u:random_normal(0,1,N)$
y:2+3·x+u$
```

Set up the log-likelihood function.

```
(%i6) loglik(alpha,beta,sigma):= −N·log(sigma) − (N/2)·log(2·%pi) −
(1/2)·sum(((y[i]−alpha−(beta·x[i]))/sigma)^2,i,1,N);
```

(%o6) $$\mathrm{loglik}(\alpha,\beta,\sigma):=(-N)\log(\sigma)-\frac{N}{2}\log(2\pi)+\left(-\frac{1}{2}\right)\sum_{i=1}^{N}\left(\frac{y_i-\alpha-\beta x_i}{\sigma}\right)^2$$

Deriviatives must be taken with respect to sigma^2, so define:
gamma == sigma^2
and take derivatives with respect to gamma.

```
(%i7) loglik(alpha,beta,gamma):= −(N/2)·log(gamma) − (N/2)·log(2·%pi) −
(1/(2·gamma))·sum((y[i]−alpha−(beta·x[i]))^2,i,1,N);
```

(%o7) $$\mathrm{loglik}(\alpha,\beta,\gamma):=\left(-\frac{N}{2}\right)\log(\gamma)-\frac{N}{2}\log(2\,\pi)+\left(-\frac{1}{2\,\gamma}\right)\sum_{i=1}^{N}(y_i-\alpha-\beta\,x_i)^2$$

Maximize it with respect to alpha, beta and gamma.

```
(%i11) sol:lbfgs(−loglik(alpha,beta,gamma),'[alpha,beta,gamma],[2.01,2.99,1.01],0.0001,[−1,0]
alfa_hat:subst(sol[1],alpha)$
beta_hat:subst(sol[2],beta)$
gmma_hat:subst(sol[3],gamma)$

(%i15) print("")$
print(alpha," = ",alfa_hat)$
print(beta," = ",beta_hat)$
print(sigma^2," = ",gmma_hat)$
```

α = 2.084553181758032
β = 2.815648702811899
σ^2 = 1.397010319508444

First-order conditions imply that:

```
(%i22) dla(alpha,beta,gamma):=''(diff(loglik(alpha,beta,gamma),alpha))$
dlb(alpha,beta,gamma):=''(diff(loglik(alpha,beta,gamma),beta))$
dls(alpha,beta,gamma):=''(diff(loglik(alpha,beta,gamma),gamma))$

foc:float(solve([
        dla(alpha,beta,gamma)=0,
        dlb(alpha,beta,gamma)=0,
        dls(alpha,beta,gamma)=0],[alpha,beta,gamma]))$

alfa_foc:subst(foc[1][1],alpha)$
beta_foc:subst(foc[1][2],beta)$
gmma_foc:subst(foc[1][3],gamma)$
```

```
(%i26) print("")$
       print(alpha," = ",alfa_foc)$
       print(beta," = ",beta_foc)$
       print(sigma^2," = ",gmma_foc)$
```

α = 2.084557555554346
β = 2.815645391810164
σ^2 = 1.397000323442532

Check to see if second-order conditions are satisfied.

```
(%i33) /* set up the Hessian matrix */

       daa(alpha,beta,gamma):=''(diff(diff(loglik(alpha,beta,gamma),alpha),alpha))$
       dab(alpha,beta,gamma):=''(diff(diff(loglik(alpha,beta,gamma),alpha),beta))$
       das(alpha,beta,gamma):=''(diff(diff(loglik(alpha,beta,gamma),alpha),gamma))$

       dbb(alpha,beta,gamma):=''(diff(diff(loglik(alpha,beta,gamma),beta),beta))$
       dbs(alpha,beta,gamma):=''(diff(diff(loglik(alpha,beta,gamma),beta),gamma))$

       dss(alpha,beta,gamma):=''(diff(diff(loglik(alpha,beta,gamma),gamma),gamma))$

       H:matrix(
         [daa(alfa_foc,beta_foc,gmma_foc),dab(alfa_foc,beta_foc,gmma_foc),das(alfa_foc,beta_f
         [dab(alfa_foc,beta_foc,gmma_foc),dbb(alfa_foc,beta_foc,gmma_foc),dbs(alfa_foc,beta_f
         [das(alfa_foc,beta_foc,gmma_foc),dbs(alfa_foc,beta_foc,gmma_foc),dss(alfa_foc,beta_fo
```

```
(%i48) print("")$
print("own−partials must be negative:")$
print("")$
print("d^2 loglik"/"(d alpha)^2"," = ",daa(alfa_foc,beta_foc,gmma_foc))$
print("")$
print("d^2 loglik"/"(d beta)^2"," = ",dbb(alfa_foc,beta_foc,gmma_foc))$
print("")$
print("d^2 loglik"/"(d gamma)^2"," = ",dss(alfa_foc,beta_foc,gmma_foc))$
print("")$
print("")$
print("the Hessian matrix:")$
print("H = ",H)$
print("")$
print("determinant of Hessian must be negative")$
print("det(H) = ",determinant(H))$
```

own−partials must be negative:

$$\frac{d\hat{}2\ loglik}{(d\ alpha)\hat{}2} = -7.15819447726231$$

$$\frac{d\hat{}2\ loglik}{(d\ beta)\hat{}2} = -5.948390501417307$$

$$\frac{d\hat{}2\ loglik}{(d\ gamma)\hat{}2} = -2.561987408715431$$

the Hessian matrix:

$$H = \begin{pmatrix} -7.15819447726231 & 0.1862052458522994 & -3.185702699150288\ 10^{-15} \\ 0.1862052458522994 & -5.948390501417307 & -6.143855205504127\ 10^{-15} \\ -3.185702699150288\ 10^{-15} & -6.143855205504127\ 10^{-15} & -2.561987408715431 \end{pmatrix}$$

determinant of Hessian must be negative
det(H) = -108.999917354472

Calculate the standard error of alpha, beta and sigma^2.

```
(%i56) info:−1·invert(H)$

print("")$
print("the information matrix:")$
print("−1·(H^−1) = ",info)$
print("")$
print(alpha," = ",alfa_foc,", se: ",sqrt(info[1,1]))$
print(beta," = ",beta_foc,", se: ",sqrt(info[2,2]))$
print(sigma^2," = ",gmma_foc,", se: ",sqrt(info[3,3]))$
```

the information matrix:

$$-1*(H\hat{}-1) = \begin{pmatrix} 0.1398138818508779 & 0.004376659238730881 & -1.843471282610332\ 10^{-1} \\ 0.004376659238730881 & 0.1682497066510845 & -4.089187041353097\ 10^{-1} \\ -1.843471282610332\ 10^{-16} & -4.089187041353097\ 10^{-16} & 0.3903219807397085 \end{pmatrix}$$

α = 2.084557555554346 , *se:* 0.3739169451240181

β = 2.815645391810164 , *se:* 0.4101825284566427

σ^2 = 1.397000323442532 , *se:* 0.6247575375613395