Part 4B Optimization

logs + exponential fnc

outline

→ OLS $\min_{\alpha,\beta} \sum \varepsilon_i^2$

→ graphical summary logs + exponential fnc
- two graphs $y = \exp(x)$ $x = \ln(y)$
- use logs to:

  $Y = K^{\alpha} L^{1-\alpha}$ ⇝ $\frac{\dot{y}}{y} = \alpha \frac{\dot{k}}{k}$
- pct changes (balanced)
- time series $d \ln Y_t$
- use log + exp in stats

  normal distribution
- take log of likelihood fn

  to work w/ addition + subtraction (simplicity)

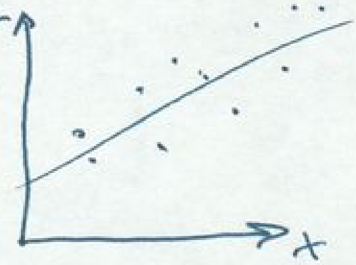
→ so need to learn
- rules of exp + ln
- rules of derivation

→ examples – binomial coeff, max likelihood

OLS

→ minimize ~~the~~ sum of squared errors

- regression



$$y_i = \alpha + \beta x_i + \varepsilon_i$$

- residual: $\varepsilon_i = y_i - \alpha - \beta x_i$

$$\min_{\alpha, \beta} \sum_{i=1}^{N} \varepsilon_i^2 = \sum_{i=1}^{N} (y_i - \alpha - \beta x_i)^2$$

- note: minimization in two parameters

First-Order Conditions

$$\frac{\partial \Sigma \varepsilon^2}{\partial \alpha} = -2 \cdot \Sigma (y - \alpha - \beta x) = 0$$

implies: $E[\varepsilon] = 0$

$$\hat{\alpha} = \bar{y} - \hat{\beta} \bar{x}$$

$$\frac{\partial \Sigma \varepsilon^2}{\partial \beta} = -2 \cdot \Sigma (y - \alpha - \beta x) x = 0$$

implies: $cor[\varepsilon, x] = 0$

$$\beta = \frac{\Sigma (y - \bar{y})(x - \bar{x})}{\Sigma (x - \bar{x})^2} = \frac{cov(x, y)}{var(x)}$$

Second-Order Conditions

$$\frac{\partial^2 \Sigma \varepsilon^2}{\partial \alpha^2} = 2 \cdot N$$

$$\frac{\partial^2 \Sigma \varepsilon^2}{\partial \beta^2} = 2 \cdot \Sigma x^2$$

own partials must be positive

(condition for a minimum)

$$\frac{\partial^2 \Sigma \varepsilon^2}{\partial \alpha \partial \beta} = 2 \cdot \Sigma x$$ ← cross partial

Hessian

$$H = \begin{bmatrix} \frac{\partial \Sigma \varepsilon^2}{\partial \alpha^2} & \frac{\partial \Sigma \varepsilon^2}{\partial \alpha \partial \beta} \\ \frac{\partial \Sigma \varepsilon^2}{\partial \alpha \partial \beta} & \frac{\partial \Sigma \varepsilon^2}{\partial \beta^2} \end{bmatrix} = \begin{bmatrix} 2N & 2\Sigma x \\ 2\Sigma x & 2 \cdot \Sigma x^2 \end{bmatrix}$$

determinant must be positive ← condition for optimum

$$|H| = 4N^2 \cdot \left(\frac{1}{N} \Sigma x^2 - \frac{1}{N} \Sigma x \frac{1}{N} \Sigma x \right)$$

$$= 4 \cdot N^2 \cdot var(x)$$

Info Matrix

$$\boxed{I = 2\cdot\sum\varepsilon^2\cdot H^{-1} = \sigma_\varepsilon^2 (X^T X)^{-1}}$$

$$I = 2\cdot\sum\varepsilon^2\cdot\frac{1}{4N^2\,var(x)}\cdot\begin{bmatrix}2\sum x^2 & 2\sum x\\ 2\sum x & 2N\end{bmatrix}$$

$$= \frac{\frac{1}{N}\sum\varepsilon_i^2}{\sum(x-\bar{x})^2}\cdot\begin{bmatrix}\frac{1}{N}\sum x^2 & -\bar{x}\\ -\bar{x} & 1\end{bmatrix}$$

$$= \frac{\sigma_\varepsilon^2}{N\cdot var(x)}\begin{bmatrix}\frac{1}{N}\sum x^2 & -\bar{x}\\ -\bar{x} & 1\end{bmatrix}$$

$$= \sigma_\varepsilon^2\left(X^T X\right)^{-1}$$

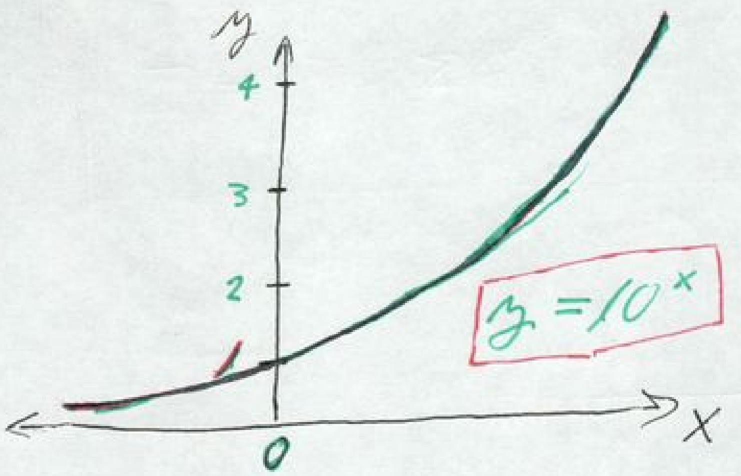
implies that

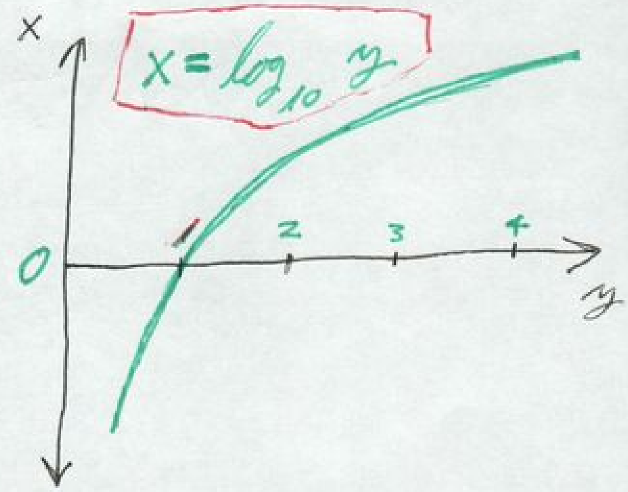
$$\sigma_\alpha = \sigma_\varepsilon\cdot\sqrt{\frac{\frac{1}{N}\sum x^2}{N\cdot var(x)}}$$

$$\sigma_\beta = \sigma_\varepsilon\cdot\sqrt{\frac{1}{N\cdot var(x)}}$$

NB: $N\cdot var(x) = \sum(x-\bar{x})^2$

Logs & Exponential Functions





→ graphs above apply to log & exponential function of any base

→ note that:

$$\boxed{\begin{array}{l} \bullet\ \log_b y = 0 \\ \bullet\ b^0 = 1 \end{array}}$$

→ idea of logarithms:

"b raised to what power equals y?"

$$10^3 = 1000 \Rightarrow \log_{10} 1000 = 3$$

The $\left\{ \begin{array}{l} \text{base-3 logarithm} \\ \text{log} \end{array} \right\}$ of 1000 $\left\{ \text{to base of 10} \right\}$ equals 3 why would you want to work w/ exponents?

because they

→ turn multiplication + division problems into addition + subtraction problems

$$\boxed{2 \cdot 4 = 8}$$

$\log_{10} 2$	$0{,}301$		$\ln 2$	$0{,}693$
$+\log_{10} 4$	$+0{,}602$		$+\ln 4$	$+1{,}386$
$\log_{10} 8$	$0{,}903$		$\ln 8$	$2{,}079$

$$2 \cdot 4 = 8$$

$$10^{0{,}301} \cdot 10^{0{,}602} = 10^{0{,}903}$$

$$e^{0{,}693} \cdot e^{1{,}386} = e^{2{,}079}$$

we can add across exponents when both of same base b

Rules of Logarithms

Rule I $\quad \log_b(A \cdot B) = \log_{10} A + \log_{10} B$

Rule II $\quad \log_b\left(\frac{A}{B}\right) = \log_b A - \log_b B$

Rule III $\quad \log_b(A^B) = \log_b\left(\prod_{i=1}^{B} A\right)$

$$= \sum_{i=1}^{B} \log_b A$$

note that
A is constant

$$\log_b(A^B) = B \cdot \log_b A$$

→ note that:

$$\ln(A+B) \neq \ln A + \ln B$$

→ logarithms of non-positive numbers are undefined preferred base

$$e \equiv \lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^m = 2{,}71828$$

$$V(m) = A\left(1 + \frac{r}{m}\right)^{mt}$$

$V \equiv$ future value
$A \equiv$ present value

where m is frequency of compounding in one year

Example:

$t = 1$ $m = 1$ $V(1) = A(1+r)^1$

$t = 1$ $m = 2$ $V(2) = A\left(1 + \frac{r}{2}\right)^2$

$m = 1$ $V(t) = A(1+r)^t$

↑ only compounding once per year

but what if interest was continuously compounded?

$$e \equiv \lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^m = 2{,}71828$$

instantaneous rate of growth

$$V = A e^{rt}$$

$$\frac{dV}{dt}\cdot\frac{1}{V} \equiv \boxed{\frac{\dot{V}}{V} = \frac{r\cdot A e^{rt}}{A e^{rt}} = r}$$

general rules of derivation

$$\frac{d}{dt} e^{f(t)} = f'(t)\, e^{f(t)}$$

$$\frac{d}{dt} \ln f(t) = \frac{f'(t)}{f(t)}$$

examples

$$f(t) = rt \implies f'(t) = r$$

$$\therefore \quad \frac{d}{dt} e^{f(t)} = \frac{d}{dt} e^{rt}$$

$$\frac{d}{dt} e^{rt} = r\cdot e^{rt}$$

examples

$$\frac{d}{dt} k \cdot \ln(at) = k \cdot \frac{d}{dt} \ln(at)$$

$$= k \cdot \frac{a}{at} = \frac{k}{t}$$

$$\boxed{Y = K^{\alpha} L^{1-\alpha}}$$

get growth rate

$$\frac{dY}{dt} = \alpha \cdot K^{\alpha-1} \cdot L^{1-\alpha} \cdot \frac{dK}{dt} + (1-\alpha) \cdot K^{\alpha} \cdot L^{-\alpha} \cdot \frac{dL}{dt}$$

$$\frac{\dot{Y}}{Y} = \alpha \cdot \frac{K^{\alpha} L^{1-\alpha}}{Y} \cdot \frac{\dot{K}}{K} + (1-\alpha) \cdot \frac{K^{\alpha} L^{1-\alpha}}{Y} \cdot \frac{\dot{L}}{L}$$

$$\boxed{\frac{\dot{Y}}{Y} = \alpha \frac{\dot{K}}{K} + (1-\alpha) \frac{\dot{L}}{L}}$$

$$\boxed{Y = K^{\alpha} L^{1-\alpha}}$$

Get growth rate

P.11

$$\ln Y = \alpha \ln K + (1-\alpha) \ln L$$

$$\frac{d}{dt} \ln Y = \frac{d \ln Y}{dY} \cdot \frac{dY}{dt} = \frac{1}{Y} \cdot \dot{Y} = \frac{\dot{Y}}{Y}$$

$$\therefore \quad \boxed{\frac{\dot{Y}}{Y} = \alpha \frac{\dot{K}}{K} + (1-\alpha) \frac{\dot{L}}{L}}$$

Maximum Likelihood

$$\mathcal{L}(\mu, \sigma^2 \mid x) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{1}{2}\left(\frac{(x_i - \mu)^2}{\sigma^2}\right)}$$

$$\ln \mathcal{L} = \frac{N}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{N} (x_i - \mu)^2$$