Lecture Three: Comparative Statics

→ comparison of different equilibrium states associated with different sets of values of parameters + exogenous variables

→ example: initial equilibrium price + qty, but then we change the value of a parameter ⇒ there will be a change in equilibrium price + qty

→ how does new equilibrium compare with the old?

- analysis may be quantitative or qualitative
- how much does endogenous variable change? OR
- does the endogenous variable increase or decrease?
- **derivative** ≡ rate of change in the equilibrium value of an endogenous variable w/ respect to rate of change in parameter or exogenous variable

→ rate of change & derivative

- we'll model rate of change in y as it responds to changes in x, where x & y are related by:

$$y = f(x)$$

y ← endogenous

x ← exogenous

- difference quotient:

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

- example

$$y = f(x) = 3x^2 - 4$$

$$\frac{\Delta y}{\Delta x} = \frac{3(x_0 + \Delta x)^2 - 4 - (3x_0^2 - 4)}{\Delta x}$$

$$= \frac{3(\cancel{x_0^2} + 2x_0\Delta x + (\Delta x)^2) \cancel{-4} - \cancel{3x_0^2} \cancel{+4}}{\Delta x}$$

$$= \frac{6x_0\Delta x + 3(\Delta x)^2}{\Delta x} = 6x_0 + 3\Delta x$$

- now consider an ~~infinitely~~ infinitesimally small change in x

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (6x_0 + 3\Delta x) = 6x_0$$

- if the limit of $\frac{\Delta y}{\Delta x}$ exists as $\Delta x \to 0$, then that limit is identified as ~~the~~ the derivative of $y = f(x)$

- primitive function $y = f(x)$
  "derived function" $\frac{dy}{dx} \equiv f'(x) \equiv \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

- because $\Delta x \to 0$ is an infinitesimally small change, the derivative is an <u>instantaneous</u> rate of change

→ the concept of limit

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{v \to 0} q$$

variation   quotient

where: $q \equiv \frac{\Delta y}{\Delta x}$ and $v \equiv \Delta x$

quotient   variation

→ what value does one variable (e.g. q) ~~the~~ approach as another variable (e.g. v) approaches a specific value (e.g. zero)?

→ more generally $v \to N$ where N is any finite real number

→ v ~~can~~ can approach N from above or below

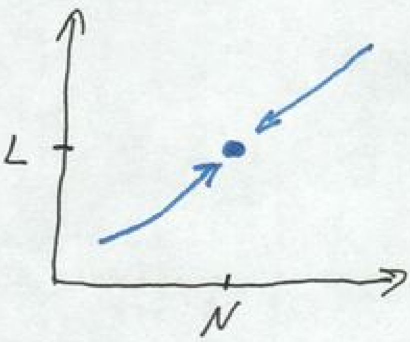
$\lim_{v \to N^+} q$ (from above)

$\lim_{v \to N^-} q$ (from below)

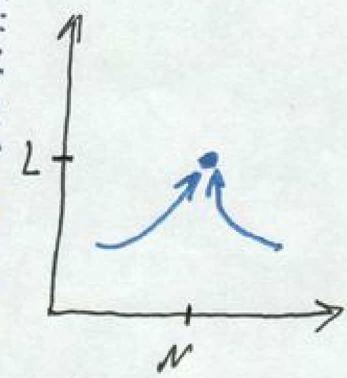
→ only when the two limits have a common finite value (e.g. L) does the limit of q exist: $\lim_{v \to N} q = L$

→ note that L must be a finite real number because if $q \to \infty$ as $v \to N$, then q does not have a limit

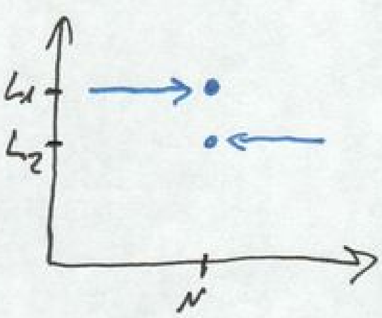
→ for convenience however, people do write: $\lim_{v \to N} q = \infty$



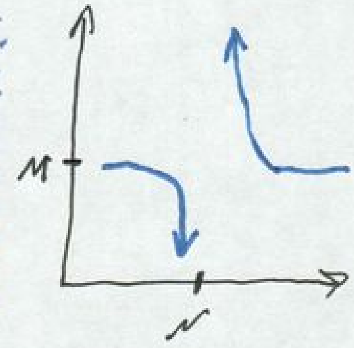
limit exists (fn is continuous at N) + fn is differentiable at N



limit exists (fn is continuous at N), but fn is NOT differentiable at N



no limit



no limit

CONTINUITY + DIFFERENTIABILITY

→ continuity requires:

- N must be in domain of the function (is $g(N)$ defined)
- $\lim_{v \to N} g(v)$ exists
- $\lim_{v \to N} g(v) = g(N)$

→ differentiability requires

- continuity (necessary)
- smoothness (sufficient)

→ use "Calculus Tricks" to cover:

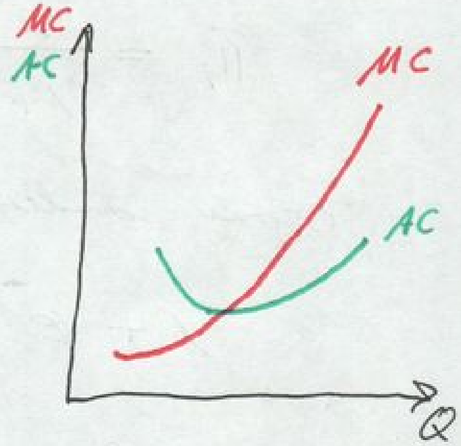
- constant function rule
- power function rule
- sum-difference rule
- product-quotient rule
- chain rule

relationship between Marginal Cost & Average Cost

$$AC \equiv \frac{TC}{Q} \qquad MC \equiv \frac{dTC}{dQ} \equiv TC'(Q)$$

$$\frac{d}{dQ}\left(\frac{TC}{Q}\right) = \frac{TC}{Q}\cdot\left(\frac{TC'(Q)}{TC} - \frac{1}{Q}\right)$$

$$= \frac{1}{Q}\cdot\left(MC - AC\right)$$



therefore:

$\frac{d}{dQ}\left(\frac{TC}{Q}\right) < 0$ AC downward sloping when $MC < AC$

$\frac{d}{dQ}\left(\frac{TC}{Q}\right) = 0$ AC minimized when $MC = AC$

$\frac{d}{dQ}\left(\frac{TC}{Q}\right) > 0$ AC upward sloping when $MC > AC$

LOGIT and LOG ODDS RATIO

$$F(\beta' x_i) \equiv \pi_i = \frac{e^{\beta' x_i}}{1 + e^{\beta' x_i}}$$

$$F^{-1}(\pi_i) = \beta' x_i = \ln\left(\frac{\pi_i}{1-\pi_i}\right)$$

because:

$$\pi_i = \frac{e^{\beta' x_i}}{1 + e^{\beta' x_i}} = \frac{1}{1 + e^{-\beta' x_i}}$$

cross multiply:

$$1 + e^{-\beta' x_i} = \frac{1}{\pi_i}$$

$$e^{-\beta' x_i} = \frac{1}{\pi_i} - \frac{\pi}{\pi} = \frac{1-\pi}{\pi}$$

$$e^{\beta' x_i} = \frac{\pi}{1-\pi}$$

$$\boxed{\beta' x_i = \ln\left(\frac{\pi_i}{1-\pi_i}\right)}$$

inverse function rule

→ if $f(x)$ is a monotonically increasing/decreasing fn of x

→ In other words, if every value of x yields a different value of y

→ then the fn f will have an inverse

function	inverse fn
$y = f(x)$	$x = f^{-1}(y)$
$y = f(x) = 5x + 25$	$x = f^{-1}(y) = \frac{1}{5}y - 5$
$\frac{dy}{dx} = 5$	$\frac{dx}{dy} = \frac{1}{5}$

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$$y = f(x) = x^3 \qquad x = f^{-1}(y) = \sqrt[3]{y} = y^{1/3}$$

$$\frac{dy}{dx} = 3x^2 \qquad \frac{dx}{dy} = \frac{1}{3x^2} = \frac{1}{3} \cdot \frac{1}{y^{1/3}} \cdot \frac{1}{y^{1/3}}$$

$$= \frac{1}{3} \cdot \frac{1}{x} \cdot \frac{1}{x}$$

SEE LOGIT EXAMPLE

Partial Differentiation

→ we often encounter function with more than one variable

$$y = f(x_1, x_2, x_3 \ldots x_n)$$

$$f_1 \equiv \frac{\partial y}{\partial x_1} \equiv \lim_{\Delta x_1 \to 0} \frac{\Delta y}{\Delta x_1}$$

→ just take the derivative with respect to x_1, holding all other variables constant

→ ex.

$$y = f(x_1, x_2) = 3x_1^2 + x_1 x_2 + 4x_2^2$$

$$f_1 \equiv \frac{\partial y}{\partial x_1} = 6x_1 + x_2$$

drops out because x_2 held constant

similarly for partial w/ respect to x_2

$$f_2 \equiv \frac{\partial y}{\partial x_2} = x_1 + 8x_2$$

→ these techniques are particularly useful when examining the effect of a change in a parameter value (comparative statics)

$$\begin{bmatrix} 1 & -1 & 0 \\ -\beta & 1 & \beta \\ -\delta & 0 & 1 \end{bmatrix} \begin{bmatrix} Y \\ C \\ T \end{bmatrix} = \begin{bmatrix} I_0 + G_0 \\ \alpha \\ \gamma \end{bmatrix}$$

$$\frac{1}{1-\beta(1-\delta)} \begin{bmatrix} 1 & 1 & -\beta \\ \beta(1-\delta) & 1 & -\beta \\ \delta & \delta & 1-\beta \end{bmatrix} \begin{bmatrix} I_0 + G_0 \\ \alpha \\ \gamma \end{bmatrix} = \begin{bmatrix} Y \\ C \\ T \end{bmatrix}$$

$$T = \frac{\delta(\alpha + I_0 + G_0) + \gamma(1-\beta)}{1-\beta(1-\delta)}$$

<u>BUDGET</u> <u>BALANCE</u>: $T - G$

$$\frac{d}{dG}(T-G) = \frac{dT}{dG} - \frac{dG}{dG}$$

$$\boxed{\begin{aligned} &-1 < \frac{d}{dG}(T-G) < 0 \\ &-1 < \frac{\delta}{1-\beta(1-\delta)} - 1 < 0 \end{aligned}}$$

$$-1 < \frac{\delta}{1-\beta(1-\delta)} - 1 < 0$$

because

$\delta < 1$

$\delta(1-\beta) < (1-\beta)$

$\delta < 1-\beta(1-\delta)$

→ National Income Model

$$Y = C + I_0 + G_0$$

$$C = \alpha + \beta(Y-T) \qquad 0<\alpha \qquad 0<\beta<1$$

$$T = \gamma + \delta Y \qquad 0<\gamma \qquad 0<\delta<1$$

$$\begin{bmatrix} 1 & -1 & 0 \\ -\beta & 1 & \beta \\ -\delta & 0 & 1 \end{bmatrix}\begin{bmatrix} Y \\ C \\ T \end{bmatrix} = \begin{bmatrix} I_0 + G_0 \\ \alpha \\ \gamma \end{bmatrix}$$

$$\begin{bmatrix} Y \\ C \\ T \end{bmatrix} = \frac{1}{1-\beta(1-\delta)}\begin{bmatrix} (I_0+G_0) + \alpha - \beta\gamma \\ (I_0+G_0)\beta(1-\delta) + \alpha - \beta\gamma \\ (I_0+G_0)\delta + (1-\beta)\gamma + \alpha\delta \end{bmatrix}$$

$$\bar{Y} = \frac{\alpha - \beta\gamma + I_0 + G_0}{1-\beta(1-\delta)}$$

now, let's look at effect of change in parameter values

Government spending multiplier

$$\frac{\partial \bar{Y}}{\partial G_0} = \frac{1}{1-\beta(1-\delta)} > 0$$

Non-income tax multiplier

$$\frac{\partial \bar{Y}}{\partial \gamma} = \frac{-\beta}{1-\beta(1-\delta)} < 0$$

Income tax-rate multiplier

$$\frac{\partial \bar{Y}}{\partial \delta} = \frac{-\beta(\alpha-\beta\gamma+I_0+G_0)}{[1-\beta(1-\delta)]^2}$$

but $\bar{Y} = \dfrac{\alpha-\beta\gamma+I_0+G_0}{1-\beta(1-\delta)}$

$$\therefore \boxed{\frac{\partial \bar{Y}}{\partial \delta} = \frac{-\beta\bar{Y}}{1-\beta(1-\delta)}} < 0$$

Partial derivatives also enable us to test for functional dependence (linear or non-linear dependence) among a set of n functions in n variables

$$y_1 = 2x_1 + 3x_2$$

$$y_2 = 4x_1^2 + 12x_1x_2 + 9x_2^2 = \underline{\underline{y_1^2}}$$

Jacobian determinant

$$|J| = \begin{vmatrix} \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_2} \\ \dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial y_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ (8x_1 + 12x_2) & (12x_1 + 18x_2) \end{vmatrix}$$

$$= 0 = (24x_1 + 36x_2) - (24x_1 + 36x_2)$$

zero because the functions are linearly dependent

Differentials

$$\Delta y \equiv \frac{\Delta y}{\Delta x} \cdot \Delta x$$

$$dy \equiv \frac{dy}{dx} \cdot dx \equiv f'(x) \cdot dx$$

$$y = 3x^2 + 7x - 5$$

$$dy = (6x + 7)dx$$

→ this is useful for obtaining elasticities

$$\varepsilon_d \equiv \frac{dQ}{dP} \cdot \frac{P}{Q} = \frac{dQ/Q}{dP/P}$$

example:

$$Q = 100 - 2P$$

$$\frac{dQ}{dP} = -2 \qquad \frac{Q}{P} = \frac{100-2P}{P}$$

$$\boxed{\varepsilon_d = -2 \cdot \frac{P}{100-2P} = \frac{-P}{50-P}}$$

Total Differentials

$$S = S(Y, i)$$

$$dS = \frac{\partial S}{\partial Y} \cdot dY + \frac{\partial S}{\partial i} di$$

$dS \equiv$ sum of changes

$dY \equiv$ change in income

$di \equiv$ change in interest rate

Total Derivatives

$y = f(x, w)$ where $x = g(w)$

$$\begin{tikzcd} w \arrow[rr, "f"] \arrow[dr, bend right, "g"'] & & y \\ & x \arrow[ur, bend right, "f"'] & \end{tikzcd}$$

Note:
direct and indirect effects of change in w on y

$$\frac{dy}{dw} = f_x \frac{dx}{dw} + f_w \frac{dw}{dw}$$

$$= \frac{\partial y}{\partial x} \cdot \frac{dx}{dw} + \frac{\partial y}{\partial w}$$

$$Q = F(K, L)$$

$$dQ = F_K \cdot dK + F_L \cdot dL$$

along isoquant $dQ = 0$

$$0 = F_K \cdot dK + F_L \cdot dL$$

$$\frac{dK}{dL} = \frac{-F_L}{F_K} \equiv \frac{-MPL}{MPK}$$

Implicit Functions

Instead of an explicit ~~fcn~~ fn $f(x) = 3x^4$

we're given: $y - 3x^4 = 0$, but

the function $y = f(x) = 3x^4$ is implicitly defined

Implicit Function Rule

→ if two expressions are identically equal, then their respective total differentials must be equal

→ example: Assume that $F(Q, K, L) = 0$ implicitly defines the production fn

$$Q = f(K, L)$$

$$MPK \equiv \frac{\partial Q}{\partial K} = \frac{-F_K}{F_Q} \qquad MPL \equiv \frac{\partial Q}{\partial L} = \frac{-F_L}{F_Q}$$

$$\boxed{\frac{\partial K}{\partial L} = \frac{-F_L}{F_K}}$$

slope of isoquant → marginal rate of ~~substitution~~ technical substitution

here Q held constant