Lecture Two (B)

Linear Models + Matrix Algebra

→ Using matrix algebra helps us write a system of equations very succinctly

→ Solving that system requires us to find the inverse, so we need a way to:

- test for the inverse of a matrix
- find the inverse (if it exists)

→ As you may recall, the matrix of coefficients MUST be square

$$\begin{bmatrix} 6 & 3 & 1 \\ 1 & 4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 22 \\ 12 \end{bmatrix}$$

not square
more variables than equations

$$\boxed{\begin{bmatrix} 6 & 3 & 1 \\ 1 & 4 & -2 \\ 4 & -1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 22 \\ 12 \\ 10 \end{bmatrix}}$$

SQUARE: Inverse may exist

$$\begin{bmatrix} 6 & 3 & 1 \\ 1 & 4 & -2 \\ 4 & -1 & 5 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 22 \\ 12 \\ 10 \\ 6 \end{bmatrix}$$

not square
more equations than variables

→ Squareness is a necessary condition for the matrix to have an inverse, BUT it is not a sufficient condition

→ the sufficient condition for the existence of an inverse is the linear independence of rows + columns

→ for example:

$\begin{bmatrix} 6 & 3 \\ 2 & 1 \end{bmatrix}$ NOT linearly independent rows (first row is 3 times the second)

$\begin{bmatrix} 6 & 3 \\ 2 & 4 \end{bmatrix}$ linearly independent rows + columns

→ It is not sufficient to have same number of equations + unknowns

→ In order to solve the system, the equations must be:

→ consistent with each other

→ functionally independent of each other

$\begin{bmatrix} 6 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 12 \\ 4 \end{bmatrix}$ $\quad$ $\begin{aligned} 6x_1 + 3x_2 &= 12 \\ 2x_1 + 1x_2 &= 4 \end{aligned}$ $\quad$ solutions: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

$\begin{bmatrix} 6 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 12 \\ 10 \end{bmatrix}$ $\quad$ $\begin{aligned} 6x_1 + 3x_2 &= 12 \\ 2x_1 + 4x_2 &= 10 \end{aligned}$ $\quad$ ONLY SOLUTION $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

→ rank of a matrix is the maximum number of linearly independent rows (or columns)

→ Determinants & Nonsingularity

→ If determinant is 0, then matrix is singular (determinantal criterion for non-singularity)

$$\begin{vmatrix} 6 & 3 \\ 2 & 1 \end{vmatrix} = 6\cdot 1 - 3\cdot 2 = 0 \quad \text{(singular)}$$

$$\begin{vmatrix} 6 & 3 \\ 2 & 4 \end{vmatrix} = 6\cdot 4 - 3\cdot 2 = 18 \quad \text{Non-singular}$$

→ Determinant of a 2×2

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

→ Determinant of a 3×3

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \underset{\substack{(+)\ i+j \\ \text{even}}}{a_{11}} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \underset{\substack{(-)\ i+j \\ \text{odd}}}{-} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \underset{\substack{(+)\ i+j \\ \text{even}}}{+} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

note pattern:

→ Note that we're expanding by alien co-factors (a_{11} is not in rows 2 or 3 and it is not in columns 2 or 3)

→ You can expand by any row or column so pick the easiest one easiest column to expand by

$$\begin{vmatrix} 5 & 6 & 1 \\ 2 & 3 & 0 \\ 7 & -3 & 0 \end{vmatrix} = 1 \cdot \begin{vmatrix} 2 & 3 \\ 7 & -3 \end{vmatrix} - \underbrace{0 \cdot \begin{vmatrix} 5 & 6 \\ 7 & -3 \end{vmatrix}}_{\text{zero}} + \underbrace{0 \cdot \begin{vmatrix} 5 & 6 \\ 2 & 3 \end{vmatrix}}_{\text{zero}}$$

$$= -2 \cdot 3 - 3 \cdot 7 = -27$$

Properties of Determinants

→ Transposition does not affect value of determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc$$

→ Interchanging rows or columns does not affect value of determinant

$$-1 \cdot \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad$$

BUT it does change its <u>sign</u>

→ multiplication by any one row or column by scalar k changes determinant's value k-fold

$$\begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = k(ad - bc)$$

→ adding/subtracting a multiple of any row to/from another row leaves value of determinant unaltered

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c+ka & d+kb \end{vmatrix} = ad + \cancel{kab} - bc - \cancel{kab}$$

→ when row/column linearly depends on another, determinant vanishes

$$\begin{vmatrix} 2a & 2b \\ a & b \end{vmatrix} = 2ab - 2ab = 0$$

$$\begin{vmatrix} 2b & b \\ 2d & d \end{vmatrix} = 2bd - 2bd = 0$$

- convenient property because it's hard to see that

$$\left.\begin{aligned} &2v_1^T - v_2^T - 3v_3^T = 0 \\ &\text{in} \quad B = \begin{bmatrix} 4 & 1 & 2 \\ 5 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} v_1^T \\ v_2^T \\ v_3^T \end{bmatrix} \end{aligned}\right\} |B| = 0$$

EXAMPLE OF MATRIX INVERSION

$$A^{-1} = \frac{1}{|A|} \cdot C^T$$

where C^T is the transpose of the co-factor matrix

$$\text{adj } A \equiv C^T$$

adjoint

when i+j is odd, the co-factor is multiplied by negative one

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$|A| = -3$$

$$A^{-1} = \frac{1}{-3} \begin{bmatrix} \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} & -1 \cdot \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} \\ -1 \cdot \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} & -1 \cdot \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} & -1 \cdot \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} \end{bmatrix}$$

$$= \frac{-1}{3} \begin{bmatrix} 2 & 1 & -7 \\ -1 & -2 & 5 \\ -1 & 1 & -1 \end{bmatrix}$$

Cramer's Rule

$$\begin{bmatrix} 6 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 12 \\ 10 \end{bmatrix}$$

$$A \quad x = d$$

$$|A| = 6 \cdot 4 - 3 \cdot 2 = 18$$

$$x_1 = \frac{1}{18} \begin{vmatrix} 12 & 3 \\ 10 & 4 \end{vmatrix} = \frac{18}{18} = 1$$

$$x_2 = \frac{1}{18} \begin{vmatrix} 6 & 12 \\ 2 & 10 \end{vmatrix} = \frac{36}{18} = 2$$

more generally

$$x_j = \frac{1}{|A|} \begin{vmatrix} a_{11} & a_{12} & \cdots & d_1 & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & d_2 & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \\ a_{n1} & a_{n2} & \cdots & d_n & \cdots & a_{nn} \end{vmatrix}$$

j-th column replaced by d

Using Cramer's Rule to solve the National Income Model

$$Y = C + I_0 + G_0 \qquad Y - C = I_0 + G_0$$

$$c = a + bY \qquad -bY + c = a$$

$$\begin{bmatrix} 1 & -1 \\ -b & 1 \end{bmatrix} \begin{bmatrix} Y \\ c \end{bmatrix} = \begin{bmatrix} I_0 + G_0 \\ a \end{bmatrix}$$

$$Y = \frac{\begin{vmatrix} I_0 + G_0 & -1 \\ a & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ -b & 1 \end{vmatrix}} = \frac{I_0 + G_0 + a}{1 - b}$$

$$C = \frac{\begin{vmatrix} 1 & I_0 + G_0 \\ -b & a \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ -b & 1 \end{vmatrix}} = \frac{a + b(I_0 + G_0)}{1 - b}$$