Lecture Two (A)

Linear Models & Matrix Algebra

purposes of matrix algebra

- → compact way of writing a system of equations
- → evaluation of determinant allows us to check for existence of a solution
- → provides method of finding solution (if it exists)

matrix algebra is only applicable to **linear** equation systems, so it

cannot work with $y = ax^b$

but it can work with $\ln y = \ln a + b \ln x$

$$\left.\begin{aligned} 6x_1 + 3x_2 + x_3 &= 22 \\ x_1 + 4x_2 - 2x_3 &= 12 \\ 4x_1 - x_2 + 5x_3 &= 10 \end{aligned}\right\} \quad \begin{bmatrix} 6 & 3 & 1 \\ 1 & 4 & -2 \\ 4 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 22 \\ 12 \\ 10 \end{bmatrix}$$

$$\boxed{A \quad x = d}$$

matrix of coefficients — vector of variables — vector of constants

A has m rows } dimension
n columns } m x n

$a_{ij} \equiv$ the element of A in the i-th row and the j-th column

column vectors are matrices with one column
row vectors are matrices with one row

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad x^T = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$$

transpose

matrix operations

addition + subtraction

$$\begin{bmatrix} 4 & 9 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 4+2 & 9+0 \\ 2+0 & 1+7 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 2 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 9 \\ 2 & 8 \end{bmatrix} - \begin{bmatrix} 4 & 9 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}$$

scalar multiplication

$$7\begin{bmatrix} 3 & -1 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 7\cdot 3 & 7\cdot -1 \\ 7\cdot 0 & 7\cdot 5 \end{bmatrix} = \begin{bmatrix} 21 & -7 \\ 0 & 35 \end{bmatrix}$$

matrix multiplication

→ the number of column in the lead matrix must equal the number of rows in the lag matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} \end{bmatrix}$$

$$(2 \times 3) \quad (3 \times 1) \quad (2 \times 1)$$

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 6 + 3 \cdot 2 + 1 \cdot 1 \\ 1 \cdot 6 + 2 \cdot 2 + 4 \cdot 1 \end{bmatrix} = \begin{bmatrix} 19 \\ 14 \end{bmatrix}$$

example

$$Y = C + I_0 + G_0$$
$$C = a + bY$$
$$\Downarrow$$
$$\boxed{\begin{aligned} Y - C &= I_0 + G_0 \\ -bY + C &= a \end{aligned}}$$

$$\begin{bmatrix} 1 & -1 \\ -b & 1 \end{bmatrix} \begin{bmatrix} Y \\ C \end{bmatrix} = \begin{bmatrix} I_0 + G_0 \\ a \end{bmatrix}$$

but how do we solve for Y & C ?

we need to <u>invert</u> the coefficient matrix i.e. just like $\frac{a}{a} = a \cdot a^{-1} = 1$

we need to find the inverse of A

$$A A^{-1} = I \equiv \text{identity matrix}$$

here's the solution (we'll explain how we arrive at this solution in a minute)

$$\underbrace{\frac{1}{1-b}\begin{bmatrix} 1 & 1 \\ b & 1 \end{bmatrix}}_{\text{inverse}} \overbrace{\begin{bmatrix} 1 & -1 \\ -b & 1 \end{bmatrix}}^{\text{original}} = \overbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}^{\text{identity}}$$

determinant

$$\frac{1}{1-b}\begin{bmatrix} 1-b & -1+1 \\ b-b & -b+1 \end{bmatrix} = \frac{1}{1-b}\begin{bmatrix} 1-b & 0 \\ 0 & 1-b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore

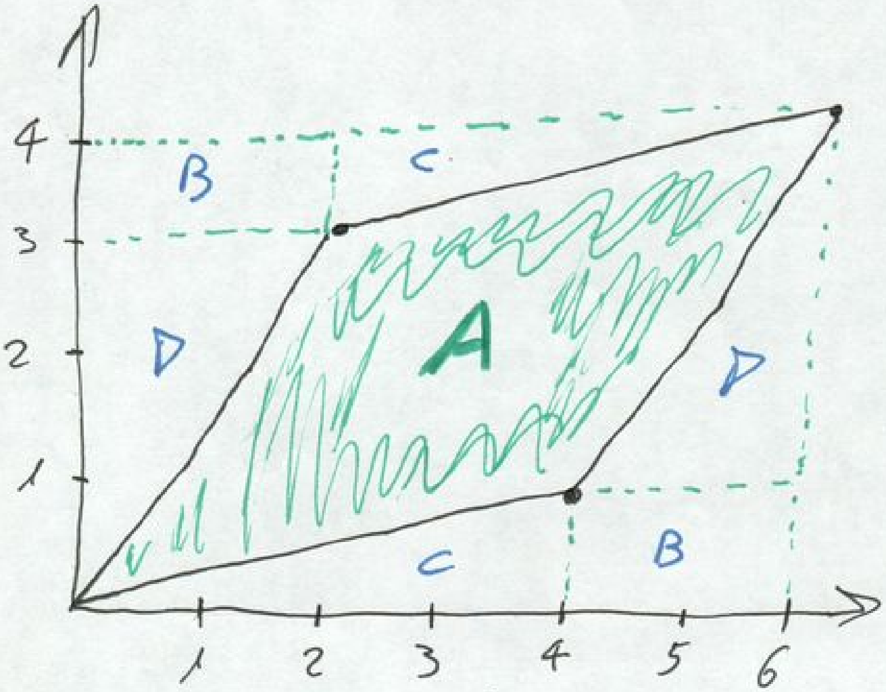
$$\frac{1}{1-b}\begin{bmatrix} 1 & 1 \\ b & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -b & 1 \end{bmatrix}\begin{bmatrix} Y \\ C \end{bmatrix} = \frac{1}{1-b}\begin{bmatrix} 1 & 1 \\ b & 1 \end{bmatrix}\begin{bmatrix} I_0 + G_0 \\ a \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} Y \\ C \end{bmatrix} = \frac{1}{1-b}\begin{bmatrix} I_0 + G_0 + a \\ b(I_0 + G_0) + a \end{bmatrix}$$

$$\begin{bmatrix} Y \\ C \end{bmatrix} = \begin{bmatrix} \frac{1}{1-b}(I_0 + G_0 + a) \\ \frac{1}{1-b}(b \cdot (I_0 + G_0) + a) \end{bmatrix}$$

The determinant of a matrix

$$A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} \qquad |A| = 4\cdot 3 - 2\cdot 1 = 10$$



total area of square is 24 — 24

area of both Bs is 2 — $-2\cdot 2 = -4$

area of both Cs is $\frac{1}{2}\cdot 4\cdot 1 = 2$ — $-2\cdot 2 = -4$

area of both Ds is $\frac{1}{2}\cdot 3\cdot 2 = 3$ — $-2\cdot 3 = -6$

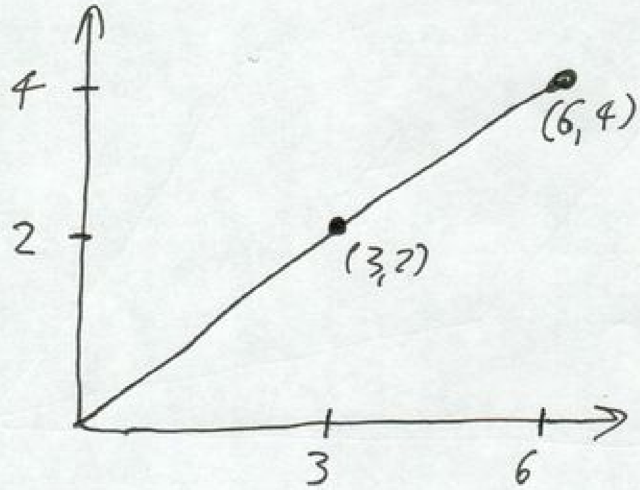
$$10$$

area of A is 10

but what if $A = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$?

the determinant is zero

$$|A| = 3 \cdot 4 - 6 \cdot 2 = 0$$



area is Zero therefore matrix is singular

before we dive into inverses, let's review

- → commutative
- → associative
- → distributive laws

because matrix algebra isn't as straightforward

Commutative law of addition

$$A + B = B + A$$

$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 6 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 3 & 6 \end{bmatrix}$$

associative law of addition

$$(A+B)+C = A+(B+C)$$

$$\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 9 \\ 1 \end{bmatrix}\right) + \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \left(\begin{bmatrix} 9 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix}\right)$$

$$\begin{bmatrix} 12 \\ 5 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 11 \\ 6 \end{bmatrix} = \begin{bmatrix} 14 \\ 10 \end{bmatrix}$$

matrix multiplication is NOT commutative

$$AB \neq BA$$

$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 6 & 2 \\ 3 & 4 \end{bmatrix} \neq \begin{bmatrix} 6 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 21 & 10 \\ 6 & 8 \end{bmatrix} \neq \begin{bmatrix} 18 & 10 \\ 9 & 15 \end{bmatrix}$$

there's a difference between pre + post multiplication however scalar multiplication is commutative

$$kA = Ak \quad \text{where } k \text{ is scalar}$$

matrix multiplication is associative

$$(AB)C = A(BC) = ABC$$

provided that the matrices are conformable

$$x^T A x = x^T (Ax) = (x^T A) x$$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_{11} x_1 \\ a_{22} x_2 \end{bmatrix} = \begin{bmatrix} x_1 a_{11} & x_2 a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x^T A x = a_{11} x_1^2 + a_{22} x_2^2$$

matrix multiplication is distributive

$$A(B+C) = AB + AC \quad \text{pre-multiplication}$$

$$(B+C)A = BA + CA \quad \text{post-multiplication}$$

identity matrices are cool because (in multiplication) they give you the multiplied matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \end{bmatrix}$$

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \end{bmatrix} = A$$

$$AI = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \end{bmatrix} = A$$

$$\underset{(m\times n)}{A} \; \underset{(n\times n)}{I} \; \underset{(n\times p)}{B} = (AI)B = \underset{(m\times n)(n\times p)}{AB}$$

null matrices (in addition) give you the added matrix

$$A + O = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = A$$

and they give you a null matrix when multiplied

$$AO = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

transpose

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \qquad B^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \qquad C^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \qquad C = C^T$$

properties

$$(A^T)^T = A$$

$$(A+B)^T = A^T + B^T$$

$$\left(\begin{bmatrix} 4 & 1 \\ 9 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 6 & 7 \end{bmatrix}\right)^T = \begin{bmatrix} 4 & 0 \\ 15 & 7 \end{bmatrix}^T = \begin{bmatrix} 4 & 15 \\ 0 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 9 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 6 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} 4 & 15 \\ 0 & 7 \end{bmatrix}$$

$$(AB)^T = B^T A^T$$

$$\left(\begin{bmatrix} 4 & 1 \\ 9 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 6 & 7 \end{bmatrix}\right)^T = \begin{bmatrix} 0 & 6 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} 4 & 9 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 3 \\ 0 & -9 \end{bmatrix}^T = \begin{bmatrix} 6 & 0 \\ 3 & -9 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 3 & -9 \end{bmatrix}$$

inverses

$$A A^{-1} = I = A^{-1} A$$

→ not every square matrix has inverse (singular matrices do not have an inverse)

→ A and A^{-1} are inverses of each other

→ square matrices only!

→ inverses are unique

$$AB = I \implies BA = I$$

now premultiply by C where $AC = CA = I$

$$CAB = CI = C$$

but because $CA = I$

$$IB = C$$

which implies that $B = C$

$\therefore$ inverses are unique

Note: $AA^{-1}B = IB = B$

but because matrix multiplication is NOT commutative ABA^{-1} is not necessarily equal to B

properties of inverses

$$(A^{-1})^{-1} = A$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$(A^{T})^{-1} = (A^{-1})^{T}$$