Lecture One — MATH METHODS

→ Mathematical vs. Literary economics

- math forces us to make assumptions explicit at each stage
- "if-then" theorems

→ Mathematical Economics vs. Econometrics

- econometrics is empirical measurement of economic data
- mathematical economics is the application of math to the theoretical aspects of economic analysis

→ this is a course in mathematical economics — use of math in deductive reasoning (not inductive study)

→ the math methods that we'll study are useful in all branches of economics (including econometrics)

→ economic model is a theoretical framework (not necessarily mathematical)

→ if mathematical

- set of $\boxed{\text{equations}}$ to describe structure of model
- relationships among $\boxed{\text{variables}}$ give mathematical form to the model's assumptions
- application of mathematical operations to ~~the~~ the equations enables us to derive $\boxed{\text{conclusions}}$ which follow logically from the assumptions

→ endogenous vs. exogenous variables

- which variables are endogenous & exogenous depends ~~on~~ on model
  - → model of price of wheat
    wheat is endogenous
  - → in theory of consumer expenditure
    price is exogenous

→ equations & identities

• ~~identity~~ definitional equation – expressions that have exactly the same meaning

$$\pi \equiv TR - TC \quad \text{(an identity)}$$

profit ≡ total revenue minus total cost

• behavioral equation – specifies the manner in which a variable behaves in response to changes in other variables

$$TC = \boxed{75} + 10Q$$
$$TC = \boxed{110} + Q^2$$

cost depends on quantity

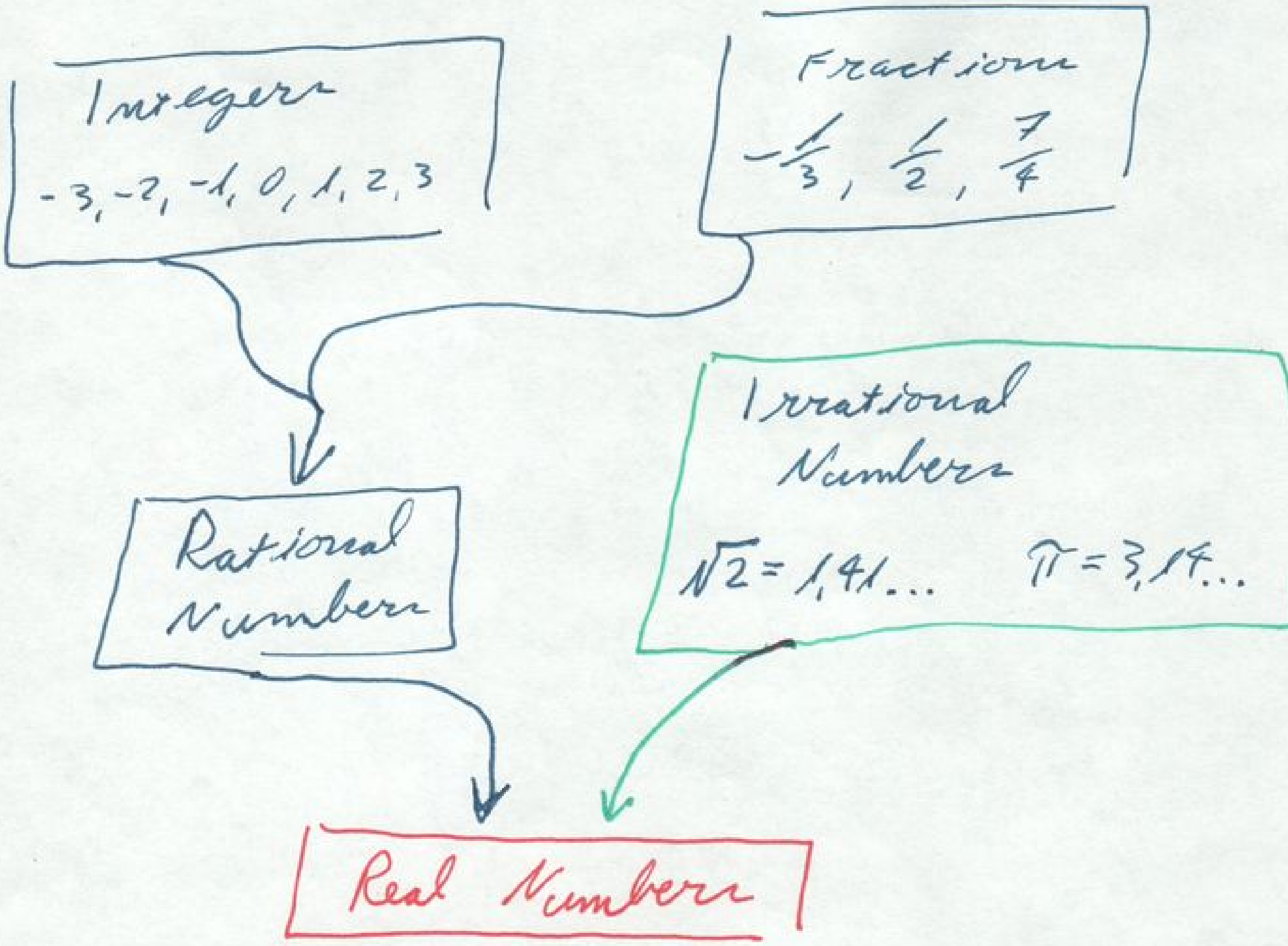
fixed cost

• equilibrium conditions – prerequisite for attaining equilibrium

$$Q_D = Q_S$$
$$S = I$$

(intended saving = intended investment)

→ the real number system



- as opposed to imaginary numbers such as $i = \sqrt{-1}$

→ concept of sets

- $S = \{2, 3, 4\}$

  S is the set of the numbers 2, 3 + 4

- $I = \{x \mid x \text{ is a positive integer}\}$

  I is the set of all positive integers

- $J = \{x \mid 2 < x < 5\}$

- elements of a set

  $3 \in S \qquad 3 \in J$

  3 is an element of set S
  3 is an element of set J

- relationships between sets

  $S_1 = \{2, 7, a, b\} \qquad S_2 = \{2, a, 7, b\}$

  $$S_1 = S_2$$

  $S = \{1, 3, 5, 7, 9\}$ and $T = \{3, 7\}$

  T is a subset of S because

  $x \in T$ implies that $x \in S$

$S = \{1, 3, 5, 7, 9\}$ $\qquad T = \{3, 7\}$

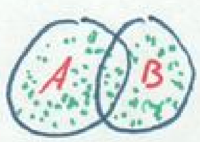
$T \subset S$ and $S \supset T$

T is contained in S $\qquad$ S includes T

- there is also the null set $\emptyset$ (contains no elements at all)

Since the null set contains no elements at all, it is a subset of ALL other sets

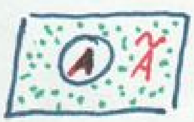
- union, intersections + complements

union  $A \cup B = \{2, 3, 4, 5, 7, 8\}$

intersection  $A \cap B = \{3\}$

$A = \{3, 5, 7\}$

$B = \{2, 3, 4, 8\}$

complement  $\tilde{A} = \{7, 8, 9\}$ ← universal set $U = \{5, 6, 7, 8, 9\}$

$A = \{5, 6\}$

$\tilde{A} = \{7, 8, 9\}$

$\tilde{A}$ is the complement of A in set U

• relations and functions

→ $\{(x, y) \mid y = 2x\}$ is a set of ordered pairs (for example: (0,0), (1,2), (-1,-2)) that constitutes a relation. On a graph, it's the set of points lying along $y = 2x$

UNIQUE RELATION

→ $\{(x, y) \mid y \leq x\}$ is also a set of ordered pairs (for example: (0,0), (1,0), (-1,-2), (-1,-1)) that constitutes a relation. On a graph it's the set of points lying along or below $y = x$ (thus satisfying $y \leq x$)

NON-UNIQUE RELATION

→ In the case of a unique relation where ~~the~~ only one y value corresponds to each x value, y is a function of x

$y = f(x)$ the function f maps (or transform) set x to set y ~~each x to y~~ $f: x \to y$

$y = f(x)$

x is the argument of function f and y is the value of the function

x is the independent variable and
y is the dependent variable

→ polynomial functions

$n=0 \qquad y = a_0$ constant fn (degenerate case)

$n=1 \qquad y = a_0 + a_1 x$ linear fn

$n=2 \qquad y = a_0 + a_1 x + a_2 x^2$ quadratic fn

$n=3 \qquad y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ cubic fn

general form:

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n$$

- quadratic function is a second-degree polynomial
cubic fn is a third-degree polynomial

- last coefficient assumed to be non-zero
otherwise would degenerate into lower-degree polynomial

→ rational functions

- the ratio of two polynomials
- ex. $y = \frac{x-1}{x^2+2x+4}$

- rectangular hyperbola

$$y = \frac{a}{x} \implies xy = a$$

when the demand curve takes this special form $Q = \frac{a}{p}$ the elasticity is constant & equal to -1 (unit elastic)

$$Q = \frac{a}{p} = a \cdot p^{-1}$$

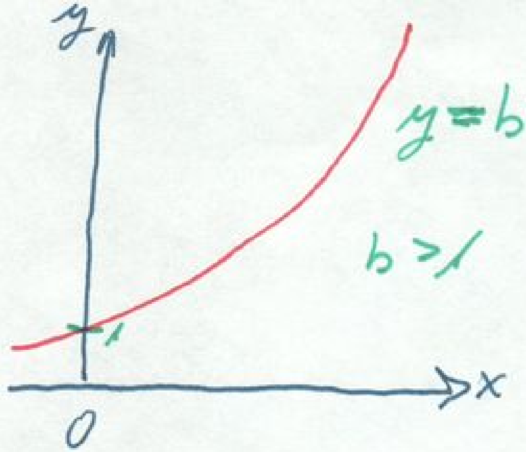
$$\frac{dQ}{dp} = -1 \cdot a \cdot p^{-2}$$

$$\frac{p}{Q} \cdot \frac{dQ}{dp} = \frac{p}{a/p} \cdot -1 \cdot a \cdot -p^{-2}$$

$$= -\frac{p^2}{a} \cdot a \cdot \frac{1}{p^2} = -1$$

→ non-algebraic functions

- exponential function $y = b^x$
- logarithmic function $y = \log_b x$



→ exponents

$$x^m \cdot x^n = x^{m+n}$$

$$\frac{x^m}{x^n} = x^{m-n} \implies \frac{1}{x^n} = x^{-n}$$

because $1 = x^0$

$$x^{1/n} = \sqrt[n]{x}$$

$$(x^m)^n = x^{mn}$$

$$x^m y^m = (xy)^m$$

→ functions of two or more independent variables

$z = g(x, y)$

$z = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + b_1 y + b_2 y^2 + \ldots + b_m y^m$

- domain of the function is no longer a set of numbers, ~~[illegible]~~
- now ~~it is~~ domain is a set of ordered pairs

we can only determine ~~[illegible]~~ z when both x and y are specified

e.1

EQUILIBRIUM ANALYSIS

→ static analysis (no effect over time)

vs.

dynamic analysis (over time)

→ here we'll focus on static analysis

→ equilibrium – "constellation of $\boxed{\text{selected}}$ $\boxed{\text{interrelated}}$ variables so adjusted to one another that no $\boxed{\text{inherent}}$ $\boxed{\text{tendency to change}}$ prevails in the model which ~~[illegible]~~ they constitute."

- selected variables – in reality, there could be many more
- interrelated – in state of rest all variables at rest
- inherent – state of rest is based on balancing of internal forces of model
- lack of tendency to change

→ equilibrium is NOT necessarily ~~[illegible]~~ ideal, e.g. a high unemployment equilibrium in the macroeconomy

EA

partial mkt equilibrium

→ three variables (simple model)

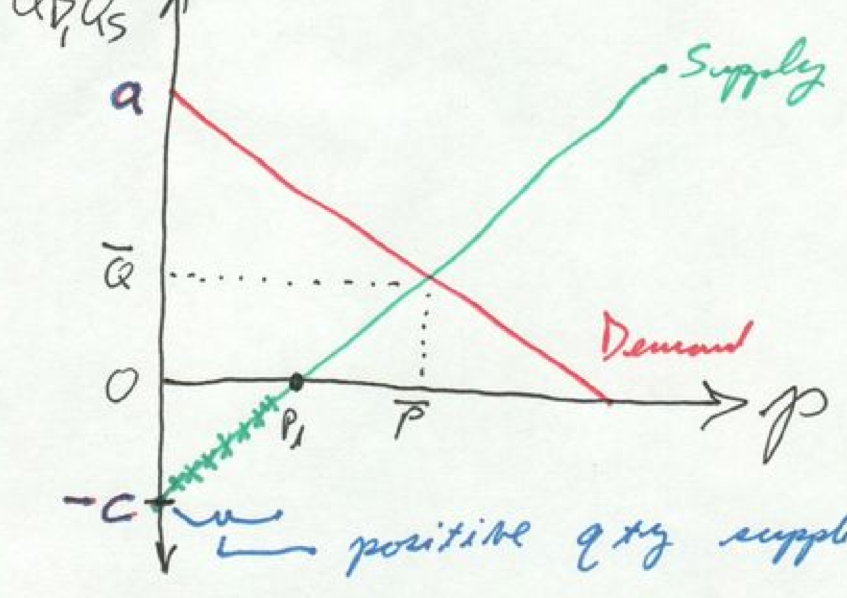
- qty demanded Q_D
- qty supplied Q_S
- price p

→ note that prices of complements + substitutes are not included here (this is partial equilibrium)

→ ASSUMPTIONS

- $Q_D = Q_S$ equilibrium condition
- $Q_D = a - bp$ $\quad Q_D$ a decreasing fn of p
- $Q_S = -c + dp$ $\quad Q_S$ an increasing fn of p

$$a, b, c, d > 0$$



solution by elimination of variables

$$Q_D = Q_S$$

$$Q_D = a - bp$$

$$Q_S = -c + dp$$

step 1:

$$a - bp = -c + dp$$

$$a + c = p(b+d)$$

$$\boxed{\bar{p} = \frac{a+c}{b+d}}$$

step 2:

$$\bar{Q} = a - \frac{b(a+c)}{b+d}$$

$$\boxed{\bar{Q} = \frac{a(b+d)}{b+d} - \frac{b(a+c)}{b+d} = \frac{da - bc}{b+d}}$$

solution by matrix algebra

$$\left.\begin{aligned} Q_D &= Q_S \\ Q_D &= a - bp \\ Q_S &= -c + dp \end{aligned}\right\} \quad \boxed{\begin{aligned} Q_D + bp &= a \\ Q_S - dp &= -c \end{aligned}}$$

$$\begin{bmatrix} 1 & b \\ 1 & -d \end{bmatrix} \begin{bmatrix} Q \\ p \end{bmatrix} = \begin{bmatrix} a \\ -c \end{bmatrix}$$

$$\begin{bmatrix} Q \\ p \end{bmatrix} = \frac{1}{-(b+d)} \begin{bmatrix} -d & -b \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ -c \end{bmatrix}$$

$$\begin{bmatrix} Q \\ p \end{bmatrix} = \frac{1}{b+d} \begin{bmatrix} da - bc \\ a + c \end{bmatrix}$$

Note: $$\frac{1}{-(b+d)} \begin{bmatrix} -d & -b \\ -1 & 1 \end{bmatrix} = \frac{1}{b+d} \begin{bmatrix} d & b \\ 1 & -1 \end{bmatrix}$$

$$\frac{1}{b+d} \begin{bmatrix} d & b \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 1 & -d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

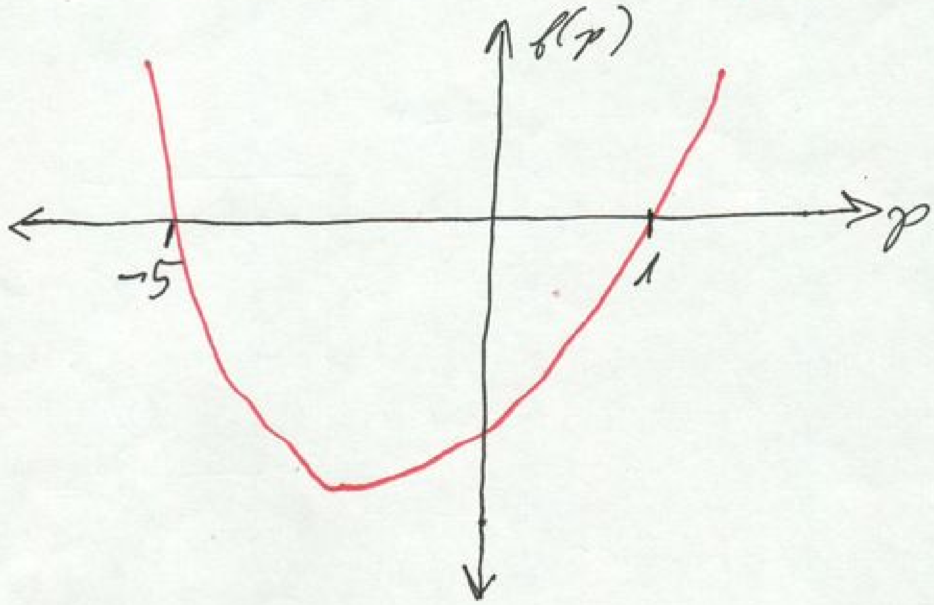
partial mkt equilibrium
non linear model

$$Q_D = Q_S$$
$$\left.\begin{aligned} Q_D &= 4 - p^2 \\ Q_S &= 4p - 1 \end{aligned}\right\}$$

$$4 - p^2 = 4p - 1$$
$$\boxed{0 = \boxed{p^2 + 4p - 5}}$$

quadratic equation: $0 = p^2 + 4p - 5$

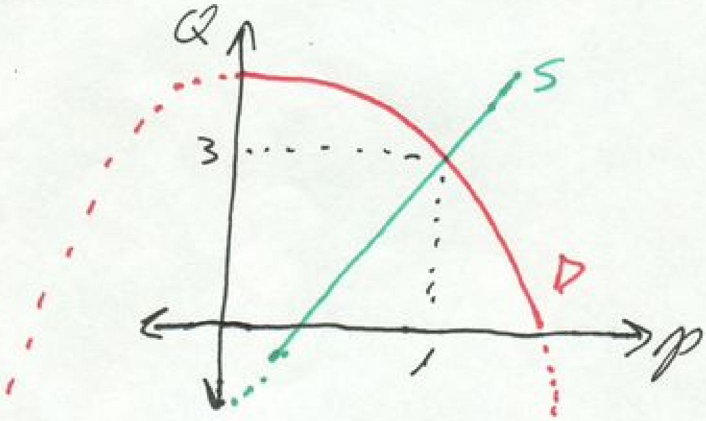
quadratic function: $f(p) = p^2 + 4p - 5$



quadratic formula: $ax^2 + bx + c = 0$

$$x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

when $p = 1$, $Q_D = Q_S = 3$



but how do we obtain quadratic formula?

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$x_1, x_2 = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

General Mkt Equilibrium

→ used in Trade Theory + in Macroeconomics

→ idea is change in price of a complement/substitute will affect demand for good

$$Q_{D,1} = Q_{S,1} \qquad Q_{D2} = Q_{S2}$$

$$Q_{D1} = a_0 + a_1 P_1 + a_2 P_2 \qquad Q_{D2} = \alpha_0 + \alpha_1 P_1 + \alpha_2 P_2$$

$$Q_{S1} = b_0 + b_1 P_1 + b_2 P_2 \qquad Q_{S2} = \beta_0 + \beta_1 P_1 + \beta_2 P_2$$

$$(a_0 - b_0) + (a_1 - b_1) P_1 + (a_2 - b_2) P_2 = 0$$

$$(\alpha_0 - \beta_0) + (\alpha_1 - \beta_1) P_1 + (\alpha_2 - \beta_2) P_2 = 0$$

Define

$$c_i \equiv a_i - b_i \qquad \forall i = 0, 1, 2$$

$$\gamma_i \equiv \alpha_i - \beta_i$$

$$c_1 p_1 - c_2 p_2 = - c_0$$

$$\gamma_1 p_1 - \gamma_2 p_2 = - \gamma_0$$

$$\bar{p}_1 = \frac{c_2 \gamma_0 - c_0 \gamma_2}{c_1 \gamma_2 - c_2 \gamma_1}$$

impose restriction

$c_1 \gamma_2 \neq c_2 \gamma_1$

$$\bar{p}_2 = \frac{c_0 \gamma_1 - c_1 \gamma_0}{c_1 \gamma_2 - c_2 \gamma_1}$$

& numerator must be of same sign as denominator for positivity

then plug in to find $\bar{Q}_1$ & $\bar{Q}_2$

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no solution $\begin{cases} x + y = 8 \\ x + y = 9 \end{cases}$

infinite number of solutions $\begin{cases} 2x + y = 12 \\ 4x + 2y = 24 \end{cases}$

unique solutions $\begin{cases} 2x + 3y = 58 \\ y = 18 \\ x + y = 20 \end{cases}$

EXAMPLE KEYNESIAN NATIONAL INCOME

$$Y = C + I_0 + G_0$$

$$C = a + bY$$

$$Y = a + bY + I_0 + G_0$$

$$\boxed{\bar{Y} = \frac{1}{1-b}\left(a + I_0 + G_0\right)}$$

$b \neq 1$

$$\bar{C} = a + b\bar{Y}$$

$$\bar{C} = \frac{a(1-b)}{1-b} + \frac{ba}{1-b} + \frac{b(I_0 + G_0)}{1-b}$$

$$\boxed{\bar{C} = \frac{a + b(I_0 + G_0)}{1-b}}$$