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Economic Growth and
Economic Fluctuations

Notes on Logarithms

When I initially designed this course, I did not plan to teach you how to use logarithms. Van den Berg's textbook however assumes that you understand logarithms, so I've written these notes to enable you to better understand the equations in his text.

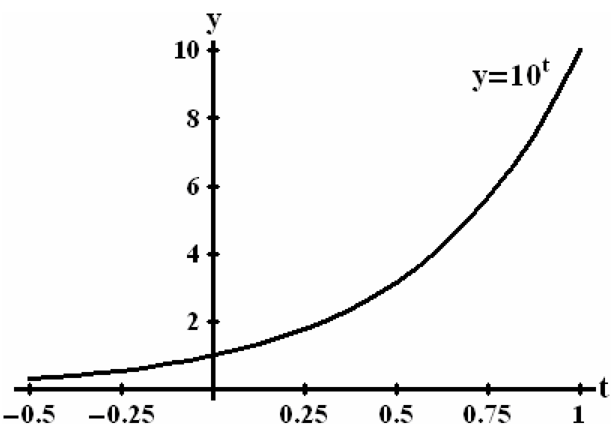
Logarithms start with a given base number. The base number can be any real number. The simplest base to use is 10, but the preferred base is the irrational number: $e = 2.71828...$. These notes explain the basic idea of logarithms using the base number 10. Then once you've grasped the basic idea behind logarithms, these notes will introduce the preferred base.

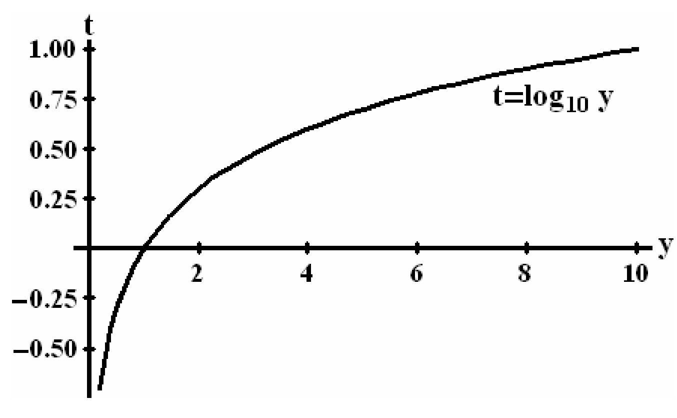
Now that we've temporarily chosen a base of 10, let's pick another number, say: 1000. The basic idea of logarithms is to answer the question: "10 raised to what power will equals 1000?" The answer of course is: "10 raised to the third power equals 1000." That is: $10^3 = 1000$. Mathematically, we say: "The logarithm of 1000 to the base of 10 equals 3." That is: $\log_{10} 1000 = 3$.

$10^3 = 1000$	$\log_{10} 1000 = 3$
$10^2 = 100$	$\log_{10} 100 = 2$
$10^1 = 10$	$\log_{10} 10 = 1$
$10^0 = 1$	$\log_{10} 1 = 0$
$10^{-1} = 0.1$	$\log_{10} 0.1 = -1$
$10^{-2} = 0.01$	$\log_{10} 0.01 = -2$
$10^{-3} = 0.001$	$\log_{10} 0.001 = -3$

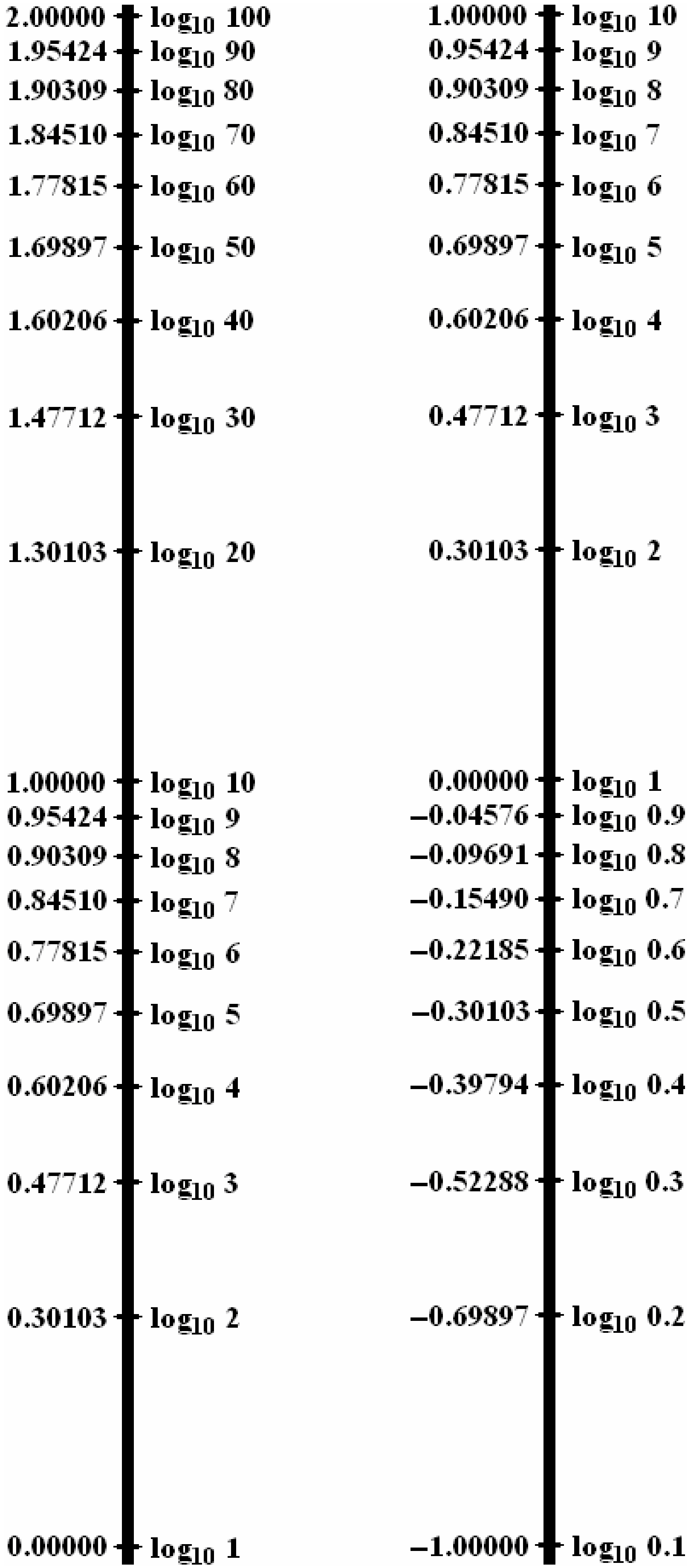
Now let's pick another number, say: 0.01 and once again ask: "10 raised to what power will equals 0.01?" The answer this time is: "10 raised to the power –2 equals 0.01." That is: $10^{-2} = 0.01$. Mathematically, we say: "The logarithm of 0.01 to the base of 10 equals –2." That is: $\log_{10} 0.01 = -2$.

This relationship is summarized in the table above and is depicted in the graphs below.





It should also be intuitively clear that if we had chosen a different base number, say: 4, then we could ask the question: "4 raised to what power equals 16?" The answer this time is: "4 raised to the second power equals 16." That is: $4^2 = 16$. Mathematically, we say: "The logarithm of 16 to the base of 4 equals 2." That is: $\log_4 16 = 2$.



Logarithms are useful because they allow us to perform the mathematical operations of multiplication and division using the simpler operations of addition and subtraction.

For example, you already know that: $2 \times 4 = 8$, so look at the logarithmic scales at left and observe that:

$$\begin{array}{rr} \log_{10} 2 & 0.30103 \\ \underline{+\log_{10} 4} & \underline{+0.60206} \\ \log_{10} 8 & 0.90309 \end{array}$$

Similarly, you know that: $\dfrac{40}{8} = 5$.
Looking again at the logarithmic scales, you can see that:

$$\begin{array}{rr} \log_{10} 40 & 1.60206 \\ \underline{-\log_{10} 8} & \underline{-0.90309} \\ \log_{10} 5 & 0.69897 \end{array}$$

In fact, before technology enabled us all to carry a calculator our pocket, people performed multiplication and division using slide rules that had base 10 logarithmic scales.

So why does this "trick" work? To answer this question, first recall that:

$$10^2 \cdot 10^3 = 10^5$$
$$100 \cdot 1000 = 100{,}000$$

$$10^2 \cdot 10^{-3} = 10^{-1}$$
$$\frac{100}{1000} = 0.1$$

So the "trick" works because the numerical value of a logarithm is an exponent and because you can add (or subtract) exponents in a multiplication problem (or division problem) so long as the exponents are the powers of a common base number.

On the previous page, we established two rules of logarithms:

$$\text{Rule I}: \quad \log_{10}(a \cdot b) = \log_{10} a + \log_{10} b$$
$$\text{Rule II}: \quad \log_{10}\left(\frac{a}{b}\right) = \log_{10} a - \log_{10} b$$

We can use Rule I to establish yet another rule:

$$\text{Rule III}: \quad \log_{10}\left(a^c\right) = c \cdot \log_{10} a$$

For example: $4^3 = 4 \cdot 4 \cdot 4$, therefore:

$$\begin{aligned} \log_{10}\left(4^3\right) &= \log_{10}(4 \cdot 4 \cdot 4) \\ &= \log_{10} 4 + \log_{10} 4 + \log_{10} 4 \\ &= 3 \cdot \log_{10} 4 \end{aligned}$$

Of course, **the rules above apply to logarithms to all bases.** After all, the numerical value of a logarithm is just an exponent and an exponent can be attached to any base number.

We've been working with logarithms to the base of 10, but in analytical work the preferred base is the irrational number: $e = 2.71828...$. Logarithms to the base of e are called **natural logarithms** (abbreviated "ln"): $\log_e a \equiv \ln a$. The rules of natural logarithms are the same as the ones derived above:

$$\text{Rule I}: \quad \ln(a \cdot b) = \ln a + \ln b$$
$$\text{Rule II}: \quad \ln\left(\frac{a}{b}\right) = \ln a - \ln b$$
$$\text{Rule III}: \quad \ln\left(a^c\right) = c \cdot \ln a$$

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pitfalls to avoid

Finally, there are two pitfalls to avoid.

First, observe from Rule I that $\ln(a+b)$ is NOT equal to $\ln a + \ln b$. Similarly, Rule II tells us that $\ln(a-b)$ is NOT equal to $\ln a - \ln b$.

Second, logarithms of non-positive numbers are undefined. For example, in the graphs on the first page, we used the equation $y = 10^t$ to obtain the relationship $t = \log_{10} y$. Therefore if $y = 0$, then the value of t must be negative infinity.

So what would the value of t be if y were a negative number? … That's a trick question. If y were a negative number, then t could not possibly be a real number. For this reason, logarithms of negative numbers are undefined.