Quantitative Analysis **Eric Doviak**
Farmingdale State College Spring 2014

Problem Set

I have designed the following set of problems to help you put the mathematical techniques that you are learning into practice. We will discuss these problems in class over the course of the semester.

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Problem #1 – A mortgage lender seeks to maximize the expected value of its portfolio. The portfolio, of course, is the sum of all of the mortgages in it, so no generality is lost by examining the case of one loan:

$$E[port] = (1-p) \cdot B + p \cdot (V - L)$$

where:

- $E[port]$ is the expected value of the portfolio
- B is the principal balance
- p is the probability of foreclosure
- $V - L$ is the amount recovered in a foreclosure sale:
  - V is the sale price at foreclosure
  - L is the legal fees incurred by foreclosure

Assume that lenders may not recover more than the principal balance through the foreclosure process. (In mathematical terms, assume that: $V - L < B$). This assumed condition was a real condition that lenders faced during the recent subprime mortgage crisis. Borrowers owed more than the homes were worth (i.e. the borrowers were "underwater") and lenders recovered far less than the principal balance.

Finally, assume that the borrower's probability of foreclosure is an increasing function of his/her "balance-to-value ratio" (i.e. B/V):

$$p \equiv p\left(\frac{B}{V}\right) \quad p' > 0$$

In other words, borrowers who are deeper underwater are more likely to enter the foreclosure process. In such cases, reducing principal balances would reduce foreclosure-related losses (by reducing the probability of foreclosure). On the other hand, principal balance reductions are a direct loss for the lender.

The following questions ask you to determine the conditions under which lenders have (or do not have) an interest in reducing principal balances. In other words, what are the conditions under which a lender would increase the expected value of its portfolio by reducing principal balances?

1. Derive the marginal benefit of reducing principal balances.
2. Derive the marginal cost of reducing principal balances.
3. What is the necessary condition for maximizing $E[port]$ with respect to the principal balance?
4. What is the sufficient condition for maximizing $E[port]$?
5. How does the marginal benefit curve shift in response to an increase in L?
6. How does the marginal cost curve shift in response to an increase in L?
7. How does the optimal principal balance change when L increases?

Problem #2 – Output per worker, y, is a function of capital per worker, k, which exhibits diminishing marginal returns. The evolution of capital per worker over time, $\dot{k} \equiv dk/dt$, is an increasing function of the saving rate, s, and a decreasing function of the depreciation rate, δ, and the labor force growth rate, n.

$$\begin{array}{llll} y = k^{\alpha} & 0 < \alpha < 1 & y \equiv y(t) & k \equiv k(t) \\ \dot{k} = s \cdot k^{\alpha} - (\delta + n) \cdot k & 0 < s < 1 & 0 < \delta < 1 & 0 < n < 1 \end{array}$$

1. Solve for the steady-state level of capital per worker, k_{ss} (i.e. the value of k when $\dot{k} = 0$).
2. Derive the marginal product of capital (i.e. $MPK \equiv dy/dk$).
3. Use the chain rule to derive the growth rate of output per worker (i.e. $\dot{y}/y$).
4. Why does the growth rate of output per worker depend on the marginal product of capital? Discuss.

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Problem #3 – The following questions ask you to derive the least squares estimates of regression coefficients. Consider the following regression model:

$$y_i = \alpha + \beta x_i + \epsilon_i$$

where ϵ_i is the error term. Rearranging terms, squaring both sides and summing over observations yields the sum of squared errors:

$$\sum \epsilon_i^2 = \sum (y_i - \alpha - \beta x_i)^2$$

The estimates of regression coefficients are the values of α and β that minimize the sum of squared errors.

1. Obtain the first-order conditions for a minimum by taking the partial derivative of $\sum \epsilon_i^2$
   (a) with respect to α
   (b) with respect to β

Define the mean of x as: $\bar{x} \equiv \frac{1}{N} \sum x_i$ and define the mean of y as: $\bar{y} \equiv \frac{1}{N} \sum y_i$.

2. Show that the first-order conditions imply that: $\widehat{\alpha} = \bar{y} - \widehat{\beta}\bar{x}$.
3. Show that the first-order conditions imply that: $\widehat{\beta} = \dfrac{\frac{1}{N} \sum x_i y_i - \bar{x}\bar{y}}{\frac{1}{N} \sum x_i^2 - \bar{x}^2}$.
4. Rearrange terms to show that: $\widehat{\beta} = \dfrac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2}$.

Next, you need to show that the second-order conditions for a minimum are satisfied.

5. Take the second partial derivative of $\sum \epsilon_i^2$
   (a) with respect to α
   (b) with respect to β
   (c) with respect to α and β (i.e. the "cross-partial")
6. Show that the second partial with respect to α is positive.
7. Show that the second partial with respect to β is positive.
8. Set up the Hessian matrix and show that its determinant is positive. (**Hint:** If you rearrange terms, you'll see that the determinant is a function of the variance of x).

Problem #4 – The following questions ask you to derive the maximum likelihood estimates of the mean and variance. Assuming that x is distributed normally with mean μ and variance σ^2, the likelihood function can be written as the product of the probability densities for each observation:

$$\pounds\left(\mu, \sigma\right) = \prod \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x_i - \mu}{\sigma}\right)^2\right)$$

1. Take the natural logarithm of the likelihood function to obtain the log-likelihood function.
2. Obtain the first-order conditions for a maximum by taking the partial derivative of the log-likelihood function:
   (a) with respect to μ
   (b) with respect to σ
3. Show that the first-order conditions imply that: $\widehat{\mu} = \frac{1}{N}\sum x_i$.
4. Show that the first-order conditions imply that: $\widehat{\sigma}^2 = \frac{1}{N}\sum\left(x_i - \mu\right)^2$.

Next, you need to show that the second-order conditions for a maximum are satisfied.

5. Take the second partial derivative of the log-likelihood function:
   (a) with respect to μ
   (b) with respect to σ^2
   (c) with respect to μ and σ^2 (i.e. the "cross-partial")
6. Show that the second partial with respect to μ is negative.
7. Show that the second partial with respect to σ^2 is negative.
8. Set up the Hessian matrix and show that its determinant is positive. (**Hint:** Make use of the fact that $\widehat{\mu} = \frac{1}{N}\sum x_i$ when the first-order conditions are satisfied).
9. Why is the estimate of a parameter's standard error smaller when the log-likelihood surface comes to a sharp peak along its dimension?