Eryk Wdowiak

Interest Rates, Growth Rates
Discounting and Time Preference

→ Suppose that you deposit \$100 in a savings account that pays 5 percent interest compounded quarterly (i.e. 4 times per year)

→ What will be the balance in?

1 quarter: $\$100 \cdot \left(1 + \frac{0{,}05}{4}\right) = \$101{,}25$

2 quarters $\$100 \cdot (1{,}0125)(1{,}0125) = \$102{,}52$

3 quarters $\$100 \cdot (1{,}0125)^3 = \$103{,}80$

4 quarters = 1 year $\$100 \cdot (1{,}0125)^4 = \$105{,}09$

5 quarters $\$100 \cdot (1{,}0125)^5 = \$106{,}41$

...

8 quarters = 2 years $\$100 \cdot (1{,}0125)^{4 \times 2} = \$110{,}45$

...

20 quarters = 5 years $\$100 \cdot (1{,}0125)^{4 \times 5} = \$128{,}20$

...

40 quarters = 10 years $\$100 \cdot (1{,}0125)^{4 \times 10} = \$164{,}36$

→ Note that the yield is different from the interest rate

- interest rate = 5 percent
- but after one year, balance = \$105,09
- the annual yield is 5,09 percent

→ More generally, let:

$A \equiv$ initial balance (deposit)

$r \equiv$ interest rate

$m \equiv$ number of times compounded per year

$t \equiv$ time since initial deposit

$V \equiv$ current balance

$$V = A\left(1 + \frac{r}{m}\right)^{mt}$$

$$\$164{,}36 = \$100\left(1 + \frac{0{,}05}{4}\right)^{4 \times 10}$$

→ Now suppose that the interest rate is 100 percent per year and the initial deposit is one dollar

→ What will be the balance in <u>one year</u> if interest is compounded?

once per year	$m = 1$	$\left(1+\frac{1}{1}\right)^1 = 2$
twice per year	$m = 2$	$\left(1+\frac{1}{2}\right)^2 = 2{,}25$
	$m = 3$	$\left(1+\frac{1}{3}\right)^3 = 2{,}37037$
	$m = 4$	$\left(1+\frac{1}{4}\right)^4 = 2{,}44141$
	$m \to \infty$	$\left(1+\frac{1}{\infty}\right)^\infty \to 2{,}7182819$

→ Infinite compounding means <u>continuous</u> compounding (as opposed to "discrete" compounding)

→ Define: $W \equiv \dfrac{m}{r}$

$$\boxed{W \cdot r \cdot t = \frac{m}{r} \cdot r \cdot t = m t}$$

$$V = A\left(1+\frac{r}{m}\right)^{mt} = A\left[\left(1+\frac{1}{w}\right)^{w}\right]^{rt}$$

$$\lim_{w \to \infty} V(w) = A * \left[\lim_{w \to \infty}\left(1+\frac{1}{w}\right)^{w}\right]^{rt}$$

$$= A \cdot (2{,}7182819)^{rt}$$

$$= A e^{rt}$$

$$V = Ae^{rt}$$

$$\boxed{e = 2{,}7182819\ldots}$$

suppose $r = 5$ percent

$t = 10$ years

$A = \$100$

$$\$164{,}87 = \$100 \cdot e^{0{,}05 \cdot 10}$$

$$= \$100 \cdot e^{0{,}5} = \$100 \cdot 1{,}6487$$

$$V = A\left(1 + \frac{r}{m}\right)^{mt}$$

suppose $m = 4$ times per year

$$\$164{,}36 = \$100 \cdot \left(1 + \frac{0{,}05}{4}\right)^{4 \cdot 10}$$

$\Rightarrow$ As m grows larger Ae^{rt} provides a better approximation to the discrete case

→ interest rate compounding is a specific example of the general process of <u>exponential growth</u>

→ r is the instantaneous rate of growth of the function Ae^{rt}

$$V = Ae^{rt}$$

change in value over time

$$\dot{V} \equiv \frac{dV}{dt} = rAe^{rt}$$

"percentage change" in value over time

$$\boxed{\frac{\dot{V}}{V}} = \frac{rAe^{rt}}{Ae^{rt}} = r$$

→ derivative of natural log of a function

$$\frac{d}{dt} \ln f(t) = \boxed{\frac{f'(t)}{f(t)}}$$

"percentage change" in value

base 10 logarithms

$\log_{10} 100 = 2$ (because $10^2 = 100$)

$\log_{10} 10 = 1$

$\log_{10} 1 = 0$

$\log_{10} \frac{1}{10} = -1$

$\log_{10} \frac{1}{100} = -2$

natural logarithms

$\log_e e^2 = \log_e 7{,}389 = 2$

$\log_e e^1 = \log_e 2{,}718 = 1$

$\log_e e^0 = \log_e 1 = 0$

define: $\ln x \equiv \log_e x$

percentage change

$$\% \Delta \text{ NYC pop} = \frac{\text{NYC pop 2010}}{\text{NYC pop 2000}} - 1$$

$$= \frac{8{,}175{,}133}{8{,}008{,}278} - 1 = 0{,}0208$$

$$\frac{\% \Delta \text{ NYC pop}}{10 \text{ years}} = \frac{0{,}0208}{10} = 0{,}00208$$

"Between 2000 and 2010, NYC population grew 2,08 percent, which corresponds to an annual growth rate of 0,208 percent" log difference = log of ratio

$$\% \Delta \text{ NYC pop} = \ln\left(\frac{\text{NYC pop 2010}}{\text{NYC pop 2000}}\right) = \ln(1{,}0208)$$

$$= \ln(\text{NYC pop 2010}) - \ln(\text{NYC pop 2000})$$

$$= \ln(8{,}175{,}133) - \ln(8{,}008{,}278)$$

$$= 0{,}0206$$

annual growth rate of NYC pop

$$n_{NYC} = 0{,}206 \text{ percent}$$

$$= 0{,}00206$$

$$V = A e^{nt}$$

$$8{,}175{,}133 = 8{,}008{,}278 \cdot e^{(0{,}00206)*10}$$

$$\text{NYC pop 2010} = \text{NYC pop 2000} \cdot e^{n_{NYC} \cdot 10 \text{ years}}$$

→ On page 1, we asked how much a \$100 deposit would grow to in 10 years

$$V = Ae^{rt}$$

$$\$164.87 = \$100\, e^{0.05 \cdot 10}$$

→ Now let's ask a different question "If I want to have \$164.87 ten years from now, how much do I have to set aside today?"

Answer: \$100

$$V \cdot e^{-rt} = A$$

$$\$164.87 \cdot e^{-0.05 \cdot 10} = \$100$$

NB: that's "e to the <u>negative</u> rt" because we <u>divided</u> both sides by e^{rt}

step 1: $V = Ae^{rt}$

step 2: $\dfrac{V}{e^{rt}} = A$

step 3: $Ve^{-rt} = A$

present value (discounting)

→ \$100 is the "present value"

\$164.87 is the "future value"

→ we "discount the future" because one dollar today is worth more than one dollar a year from now because today's dollar can "earn interest in a savings account"

$$Ve^{-rt} = A \qquad V\left(1 + \frac{r}{m}\right)^{-mt} = A$$

time preference

→ at 5 percent interest, a dollar one year from now is only worth 95 cents today

→ but suppose that I am impatient

(I want to consume now. I don't to put 95 cents aside now, so that I will have one dollar next year).

→ suppose my rate of time preference is 10 percent (I am willing to set 90 cents aside, so that I will have one dollar next year).

→ interest rate $r = 5\%$

rate of time preference $\rho = 10\%$

→ in this scenario, I am more impatient than the economy as a whole. I am more interested in present consumption than the economy as a whole, so

→ I borrow from people ~~who~~ who are more patient than me. (The nice people who opened a savings deposit at the community bank).